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Auxiliary matrices for the six-vertex model at $q^N = 1$ and a geometric interpretation of its symmetries

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Abstract

The construction of auxiliary matrices for the six-vertex model at a root of unity is investigated from a quantum group theoretic point of view. Employing the concept of intertwiners associated with the quantum loop algebra $U_q(\widetilde{sl}_2)$ at $q^N = 1$, a three-parameter family of auxiliary matrices is constructed. The elements of this family satisfy a functional relation with the transfer matrix allowing one to solve the eigenvalue problem of the model and to derive the Bethe ansatz equations. This functional relation is obtained from the decomposition of a tensor product of evaluation representations and involves auxiliary matrices with different parameters. Because of this dependence on additional parameters, the auxiliary matrices break in general the finite symmetries of the six-vertex model, such as spin-reversal or spin-conservation. More importantly, they also lift the extra degeneracies of the transfer matrix due to the loop symmetry present at rational coupling values. The extra parameters in the auxiliary matrices are shown to be directly related to the elements in the enlarged centre Z of the algebra $U_q(\widetilde{sl}_2)$ at $q^N = 1$. This connection provides a geometric interpretation of the enhanced symmetry of the six-vertex model at rational coupling. The parameters labelling the auxiliary matrices can be interpreted as coordinates on a hypersurface Spec $Z \subset \mathbb{C}^4$ which remains invariant under the action of an infinite-dimensional group G of analytic transformations, called the quantum coadjoint action.

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1. Introduction

Almost 40 years ago Lieb [1–3] and Sutherland [4] solved the six-vertex or XXZ model associated with the following quantum spin-chain Hamiltonian:

$$H = \sum_{m=1}^{M} \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \frac{q+q^{-1}}{2}\left(\sigma_m^z \sigma_{m+1}^z - 1\right) \qquad \sigma_{M+1}^{x,y,z} \equiv \sigma_1^{x,y,z}. \tag{1}$$

Here σ_m^x, σ_m^y and σ_m^z are the Pauli matrices acting on the mth site of the spin-chain. Applying the coordinate space Bethe ansatz [5], the eigenvalues of the Hamiltonian (1) and the associated transfer matrix (see equation (3)) can be determined by the solutions of the following set of equations:

$$\left\{\frac{\sinh\frac{1}{2}\left(u_j^B + \mathrm{i}\gamma\right)}{\sinh\frac{1}{2}\left(u_j^B - \mathrm{i}\gamma\right)}\right\}^M = \prod_{\substack{l=1\\ l\neq j}}^{n_B} \frac{\sinh\frac{1}{2}\left(u_j^B - u_l^B + 2\mathrm{i}\gamma\right)}{\sinh\frac{1}{2}\left(u_j^B - u_l^B - 2\mathrm{i}\gamma\right)} \qquad q = \mathrm{e}^{\mathrm{i}\gamma} \quad \gamma \in \mathbb{R}. \tag{2}$$

The integer n_B is related to the total spin of the associated eigenvector. Throughout this paper, I shall refer to the finite solutions u_j^B as Bethe roots and to (2) as the Bethe ansatz equations.

In recent years, the Bethe ansatz equations and the algebraic structure of the six-vertex model at roots of unity, $q^N = 1$, have again been the focus of discussion (see, e.g., [6–11]). These new developments address the nature of the extra degeneracies in the spectrum of the six-vertex transfer matrix and (1) at a root of unity. The discovery of these degeneracies originated in Baxter's papers on the eight-vertex model [16–18], where the six-vertex limit can be taken. Thirty years later, Deguchi *et al* showed that there is an infinite-dimensional symmetry underlying these degeneracies [6]. In the commensurate sectors where the total spin is a multiple of the order *N*, the transfer matrix and thus the Hamiltonian (1) can be shown to be invariant under the action of the loop algebra $\widetilde{sl}_2 = sl_2 \otimes \mathbb{C}[t, t^{-1}]$. Although the present discussion is limited to the six-vertex model with spin one-half, the occurrence of the loop symmetry at roots of unity is a general phenomenon. See [12, 13] for generalizations to models with higher spin and higher rank.

Fabricius and McCoy pointed out [7, 9] that in order to understand the structure of the degenerate eigenspaces and the enhanced symmetry at rational coupling, it is not sufficient to look at the solutions of the Bethe ansatz equations (2) alone. In fact, the loop symmetry shows that the degenerate eigenspaces of (1) at $q^N = 1$ contain linear combinations of states whose total spin differs by multiples of *N*. Hence, the coordinate space Bethe ansatz, which starts from the assumption of spin-conservation, is not suitable for analysing the full symmetry present at roots of unity. An alternative approach which solves the eigenvalue problem of integrable models such as (1) and does not rely on the conservation of the total spin is the concept of auxiliary matrices. This technique was first introduced by Baxter in connection with his solution to the eight-vertex model [14–18]. His method will be briefly described in section 1.2.

In this paper, auxiliary matrices for the six-vertex model at roots of unity will be constructed which differ from Baxter's. The differences will be explained in detail in section 1.3. The construction procedure and the assumptions made on the form of the auxiliary matrices in this work are motivated by the known results on root-of-unity representations of the quantum loop algebra $U_q(\widetilde{sl}_2)$ and its finite counterpart $U_q(sl_2)$ [20–22].

The main result of this work is the derivation of a functional relation (see equation (24)) between the auxiliary matrices and the six-vertex transfer matrix from representation theory. All operators in this functional equation are proved to commute with each other which allows one to derive the Bethe ansatz equations (2) and to determine the spectrum of the transfer matrix. This will be explicitly demonstrated for two examples. It is important to note that this functional equation is different from the one considered by Baxter (see equation (18)). The functional equation derived in this paper involves three different auxiliary matrices instead of only a single one.

In addition, the auxiliary matrices are shown to break the finite and the infinite-dimensional symmetries of the six-vertex model at roots of unity due to the dependence on special parameters. These parameters encode a rich geometric structure which makes the auxiliary matrices mathematically interesting objects for further studies. At the end of this paper, the action of an infinite-dimensional automorphism group will be defined on the auxiliary matrices using the results in [20, 21]. Since all auxiliary matrices will be shown to commute with the six-vertex transfer matrix, while they in general do not commute among themselves, this group action manifests the infinite-dimensional non-Abelian symmetry of the six-vertex model at $q^N = 1$. Recall that the loop symmetry has only been established in the commensurate sectors where the total spin is a multiple of N [6, 12, 13]. In contrast, the auxiliary matrices will be defined for all spin-sectors.

The focus of this paper is mainly on the construction of the auxiliary matrices, their properties and their geometric description. The implications for the analysis of the degenerate eigenspaces of the six-vertex model will be subject to future investigations [23].

As the discussion of the degeneracies at roots of unity and the construction of auxiliary matrices involves various technical subtleties, it is worthwhile first presenting the definition of the six-vertex model. This will enable us to make precise statements with regard to the different approaches of constructing auxiliary matrices. It will also allow us to briefly review recent developments concerning the degeneracies at roots of unity. The connection with the literature [16, 19, 24–27] concerning previous results on auxiliary matrices for the six-vertex model will be made in section 1.4.

1.1. The six-vertex model, definitions and conventions

Consider an $M \times M'$ square lattice with the partition function written in terms of the transfer matrix T as

$$Z(z) = \operatorname*{Tr}_{(\mathbb{C}^2)^{\otimes M}} T(z)^{M'} \qquad T(z) = \operatorname*{Tr}_0 R_{0M}(z) R_{0M-1}(z) \cdots R_{01}(z). \tag{3}$$

Here $R = R(z,q)$ is a matrix defined over $\mathbb{C}^2 \otimes \mathbb{C}^2$ and contains the Boltzmann weights associated with the different vertex configurations,

$$R = \frac{a+b}{2} 1 \otimes 1 + \frac{a-b}{2} \sigma^z \otimes \sigma^z + c\sigma^+ \otimes \sigma^- + c'\sigma^- \otimes \sigma^+ = \begin{pmatrix} a & & & \\ & b & c & \\ & c' & b & \\ & & & a \end{pmatrix}. \tag{4}$$

The above matrices are defined as $\sigma^+ = (\sigma^x + \mathrm{i}\sigma^y)/2, \sigma^- = (\sigma^x - \mathrm{i}\sigma^y)/2$ and the lower indices in (3) indicate on which pair of spaces the R-matrix acts in the $(M+1)$-fold tensor product of $\mathbb{C}^2$. The Boltzmann weights of the six allowed vertices can be parametrized as follows:

$$a = \rho \qquad b = \rho \frac{(1-z)q}{1-zq^2} \qquad c = \rho \frac{1-q^2}{1-zq^2} \qquad c' = cz \qquad z = \mathrm{e}^u q^{-1} \in \mathbb{C}^\times. \tag{5}$$

The function $\rho = \rho(z,q)$ is an arbitrary normalization factor which might depend on the spectral parameter z and the deformation parameter q . Setting $\rho(z=1,q) = 1$, the matrix (4) becomes the permutation operator at $z = 1$ and the transfer matrix reduces to the shift operator. The associated XXZ spin-chain Hamiltonian (1) is obtained by taking the logarithmic derivative of the transfer matrix (3) with respect to the spectral parameter. As is well known, the six-vertex model is integrable as the transfer matrix evaluated at different spectral parameters commutes with itself, $[T(z), T(w)] = 0$. Further, symmetries are given by the conservation of the total spin

$$[T(z), S^z] = 0 \qquad S^z = \frac{1}{2}\sum_{m=1}^{M} \sigma_m^z \tag{6}$$

and by the commutation of the transfer matrix with the two idempotent operators

$$\mathfrak{R} = \sigma^x \otimes \cdots \otimes \sigma^x \qquad \mathfrak{S} = \sigma^z \otimes \cdots \otimes \sigma^z = (-1)^{M/2-|S^z|}. \tag{7}$$

The first operator invokes spin-reversal while the second has eigenvalue +1 or −1 depending whether the number of down spins n in a state is even or odd. The operators $\mathfrak{R}$, $\mathfrak{S}$ can be used to derive for spin-chains of even length the useful relations,

$$M \text{ even}: T(z, q^{-1}) = T(z^{-1}, q) \qquad \text{and} \qquad T(z, -q) = \mathfrak{S}T(z, q) = T(z, q)\mathfrak{S}. \tag{8}$$

Because of the degeneracies at roots of unity, we will not analyse the structure of the six-vertex model via the coordinate space Bethe ansatz (see, e.g., [10] for a recent discussion). Instead, we employ the method of auxiliary matrices and functional equations, which is described in the following section.

1.2. Baxter's auxiliary matrix and functional equation

In 1971, Baxter noted that the Bethe ansatz equations (2) for the six-vertex model ensure the existence of an auxiliary matrix Q subject to the following functional relation with the transfer matrix [14]:

$$T(z)Q(z) = Q(z)T(z) = b(z)^M Q(zq^2) + a(z)^M Q(zq^{-2}). \tag{9}$$

In addition, Baxter postulated (see section 9.5, page 184 in [19]) that an auxiliary matrix ought to obey the following commutation relations:

$$[T(z), Q(w)] = 0 \tag{10}$$

$$[Q(z), Q(w)] = 0 \tag{11}$$

$$[Q(z), \mathfrak{S}] = 0 \qquad z, w \in \mathbb{C}. \tag{12}$$

For $w = z, zq^{\pm 2}$, the first two commutators imply that all matrices in the functional equation (9) can be simultaneously diagonalized. Hence, all the eigenvalues of the transfer matrix (3) can be expressed in terms of those of an auxiliary matrix $Q(z)$ provided the latter is non-singular. Imposing the commutation relations (10) and (11) for arbitrary values of w guarantees that the eigenvectors can be chosen independent of the spectral parameter z. The last commutator (12) imposes invariance under the transformation $q \to -q$ (cf equation (8)). There is no mentioning made in [19] with respect to the behaviour of the auxiliary matrix under spin-reversal.

The Bethe ansatz equations (2) are recovered whenever the eigenvalue of $Q(z)$ vanishes for some value $z = z_j^B$ while the corresponding eigenvalues of $Q\big(z_j^B q^2\big)$, $Q\big(z_j^B q^{-2}\big)$ are non-zero,

$$0 = Q\big(z_j^B\big)T\big(z_j^B\big) = a\big(z_j^B\big)^M Q\big(z_j^B q^{-2}\big) + b\big(z_j^B\big)^M Q\big(z_j^B q^2\big). \tag{13}$$

Here the eigenvalues and the corresponding matrices are denoted by the same symbol. The zeroes $z_j^B = \mathrm{e}^{u_j^B} q^{-1}$ coincide with the Bethe roots in (2). Note that there might be further zeroes z_j for which all three eigenvalues in (13) simultaneously vanish. This is of crucial importance for the case when q is a root of unity.

1.2.1. Degeneracies and complete N-strings at $q^N = 1$. Suppose we are given a solution $Q(z)$ to Baxter's equation (9) which satisfies (10)–(12) and lifts the degeneracy in the eigenspaces of the six-vertex transfer matrix. The functional equation then implies for even roots of unity, $N = 2N'$, that the eigenvalues of $Q(z)$ must contain additional factors of the form [7, 18]

$$Q_{N'}(z, z_0) = \prod_{\ell=0}^{N'-1} (z - z_0 q^{2\ell}) = z^{N'} - z_0^{N'} \qquad q^{2N'} = 1. \tag{14}$$

These factors amount to the existence of complete N'-strings $(z_0, z_0 q^2, \ldots, z_0 q^{2N'-2})$ first observed in the context of the eight-vertex model [18]. As has been argued by Fabricius and McCoy [7] the string centre z_0 is not fixed by the Bethe ansatz equations (2) since the factors (14) drop out of the functional equation (9) due to the obvious periodicity

$$Q_{N'}(z, z_0) = Q_{N'}(zq^2, z_0). \tag{15}$$

Note that the string centre z_0 is in fact only determined up to multiplication by q^2, whence the factors (14) depend on $z_0^{N'}$ rather than z_0. For odd roots of unity, the period of the complete strings is given by q rather than q^2.

Consequently, the Bethe ansatz equations (2) alone are not sufficient to describe the degenerate eigenspaces of the transfer matrix [7]. One ought to construct a consistent auxiliary matrix which lifts the degeneracy of the six-vertex transfer matrix at roots of unity by fixing the string centres z_0. In [10] (see the comment on p 25, after equation (94)), Baxter argued that the arbitrariness in choosing z_0 should allow for the existence of a one-parameter family of auxiliary matrices at a root of unity.

We shall see in section 5 of this paper that for N odd there even exists a three-parameter family provided the conditions (11) and (12) are dropped.

1.2.2. Baxter's construction of auxiliary matrices. In Baxter's approach to the eight- [15–19] and six-vertex model [16, 19], two 'preliminary' auxiliary matrices $Q_{R,L}(z)$ are introduced both of which satisfy the functional equation (9) and are of the following form:

$$Q(z) = \operatorname*{Tr}_0 L_{0M}(z/\mu) L_{0M-1}(z/\mu) \cdots L_{01}(z/\mu) \qquad L_{0m} \in \operatorname{End}(V_0 \otimes V_m). \tag{16}$$

Here $\mu \in \mathbb{C}$ is a possible scaling factor, V_0 denotes the auxiliary space and the tensor product of the vector spaces $V_m \cong \mathbb{C}^2$, $1 \leqslant m \leqslant M$ forms the quantum spin-chain. Neither of the matrices Q_R, Q_L commutes with the transfer matrix. The final auxiliary matrix which commutes with the transfer matrix is given by

$$Q_B(z) = Q_R(z) Q_R(z_R)^{-1} = Q_L(z_L)^{-1} Q_L(z). \tag{17}$$

Here $z_{R,L}$ are some arbitrary reference points at which the matrices $Q_{R,L}$ are supposed to be non-singular. In general, the auxiliary matrix is therefore not of the simple form (16) which makes it unwieldy in light of algebraic manipulations.

In the case of the six-vertex model, an explicit formula for an auxiliary matrix was given by Baxter only for the sectors of vanishing total spin [16],

$$S^z = 0: \quad Q(z)^{\beta_1 \cdots \beta_M}_{\alpha_1 \cdots \alpha_M} = \mathcal{N}_\infty \exp\left(\frac{1}{4} \mathrm{i}\gamma \sum_{m=1}^{M} \sum_{n=1}^{m-1} (\alpha_n \beta_m - \alpha_m \beta_n) + \frac{1}{4} u \sum_{m=1}^{M} \alpha_m \beta_m \right). \tag{18}$$

Here $\alpha_m, \beta_m = \pm 1$ are the eigenvalues of σ^z at the mth site. This expression ought to hold for all values of $\gamma \in \mathbb{R}$ and spin-chains of even length.

1.3. Construction of auxiliary matrices via quantum groups

In this paper, a different approach will be used which requires that the final auxiliary matrix be of the simple form (16) (see also, e.g., [25, 27]). This assumption has several consequences for the choice of the L-matrix used in (16) and for the form of the functional equation with the transfer matrix which will turn out to be different from Baxter's equation (9).

In order to satisfy (10), one now demands that the R- and L-matrix obey the Yang–Baxter equation [28, 29]

$$L_{12}(w/z)L_{13}(w)R_{23}(z) = R_{23}(z)L_{13}(w)L_{12}(w/z). \tag{19}$$

In the context of trigonometric integrable vertex models, the solutions to this equation can be classified through intertwiners associated with quantum groups. The latter are non-cocommutative Hopf algebras introduced by Drinfel'd [30] and Jimbo [31]. The algebraic structure of quantum groups is intimately linked with the quantum inverse scattering method of the Faddeev school [32–34].

The simplest example of an intertwiner is the six-vertex R-matrix (4),

$$R(z)\left(\pi_z^0 \otimes \pi_1^0\right)\Delta(x) = \left[\left(\pi_z^0 \otimes \pi_1^0\right)\Delta^{\mathrm{op}}(x)\right]R(z) \qquad x \in U_q(\widetilde{sl}_2). \tag{20}$$

Here π_z^0 denotes the two-dimensional evaluation representation of the quantum loop algebra $U_q(\widetilde{sl}_2)$,

$$\begin{aligned} &\pi_z^0(e_0) = z\sigma^- \qquad &&\pi_z^0(f_0) = z^{-1}\sigma^+ \qquad &&\pi_z^0(k_0) = q^{-\sigma^z} \\ &\pi_z^0(e_1) = \sigma^+ \qquad &&\pi_z^0(f_1) = \sigma^- \qquad &&\pi_z^0(k_1) = q^{\sigma^z}. \end{aligned} \tag{21}$$

The symbols Δ, Δ^{op} stand for the coproduct and opposite coproduct whose definition will be given in the text (see equations (28) and (51)). The symbols $\{e_i, f_i, k_i\}_{i=0,1}$ denote the Chevalley–Serre generators of $U_q(\widetilde{sl}_2)$.

In the present construction of auxiliary matrices, the L-operator will also be defined as an intertwiner but with the representation (21) in the first factor replaced by a more general representation of the quantum loop algebra at a primitive root of unity, $q^N = 1$ with $N \geqslant 3$. That is, the L-matrix in (16) has to obey the relation

$$L^p(w/z)\left(\pi_w^p \otimes \pi_z^0\right)\Delta(x) = \left[\left(\pi_w^p \otimes \pi_z^0\right)\Delta^{\mathrm{op}}(x)\right]L^p(w/z) \qquad x \in U_q(\widetilde{sl}_2) \tag{22}$$

where $\pi_w^p : U_q(\widetilde{sl}_2) \to \mathrm{End}\, V_0 \cong \mathrm{End}\, \mathbb{C}^N$ is some finite-dimensional irreducible evaluation representation. We will see explicit examples in section 2 (see definition (43)). However, since the trace is taken in (16), the explicit form of the representation π_w^p only matters up to isomorphism. As will be explained in the text, one may use the results in [20, 21] to show that all isomorphic representations can be labelled in terms of points p on a three-dimensional complex hypersurface Spec $Z \subset \mathbb{C}^4$. For example, when N is odd the points on the hypersurface obey

$$p = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{c} = \mu + \mu^{-1}) \in \mathbb{C}^4 \qquad \mathbf{xy} + \mathbf{z} + \mathbf{z}^{-1} = \mu^N + \mu^{-N}. \tag{23}$$

The structure of Spec Z is connected with the algebraic properties of the centre Z of the algebra $U_q(\widetilde{sl}_2)$ (see definition (36) in section 2 of this paper). The parameter $\mu \neq 1$ is identical with the scaling factor in (16).

Thus, while there are many different ways of writing down solutions to (19) respectively (22) the final auxiliary matrices $\{Q_p(z)\}$ (defined in section 5, equation (106)) only depend on the point $p \in \mathrm{Spec}\, Z$. The choice of a representation π_w^p corresponds to a choice of coordinates on the hypersurface and does not effect the final form of the auxiliary matrix $Q_p(z)$.

The second step in the construction is to find a functional relation with the transfer matrix (3) analogous to (9). In fact, we will see that due to the different assumptions made in comparison to [19], the family of the auxiliary matrices obeys a functional equation of a more general form than (9), namely

$$Q_p(z)T(z) = b(z)^M Q_{p'}(zq^2) + a(z)^M Q_{p''}(zq^{-2}). \tag{24}$$

The points p' and p'' are determined via an exact sequence which describes the decomposition of the tensor product $\pi_w^p \otimes \pi_z^0$ when the ratio z/w is fixed to the value μ in (23),

$$0 \to \pi_{w'}^{p'} \hookrightarrow \pi_w^p \otimes \pi_z^0 \to \pi_{w''}^{p''} \to 0. \tag{25}$$

Here $\pi_{w'}^{p'}, \pi_{w''}^{p''}$ denote some other irreducible root-of-unity representations of $U_q(\widetilde{sl}_2)$ which will be explicitly calculated.

In contrast to (9), the three auxiliary matrices appearing in (24) have different spectra even when they are evaluated at the same value of the spectral variable z. That is, the eigenvalues of $Q_p(z)$, $Q_{p'}(z)$, $Q_{p''}(z)$ in general do not coincide. This is the major difference with expression (18) and Baxter's construction. Now one has to allow for a shift in the additional parameters. Furthermore, for generic points $p \in \operatorname{Spec} Z$, the auxiliary matrices Q_p constructed in this paper violate (11) and (12). However, those restrictions are in general too strong: the minimal requirement to derive the eigenvalues of the transfer matrix (3) and the Bethe ansatz equations (2) from (24) is to demand that all matrices in the functional equation commute with each other.

Note that by defining the *L*-matrix as a solution to (19), we have also imposed the overly restrictive condition (10). The motivation for this definition is the connection with representation theory via (22).

Despite the described differences between the two approaches of constructing auxiliary matrices, we will see in section 5 that solutions to Baxter's functional equation (9) can be obtained by taking a finite sum over the solutions to (24) (cf equations (117) and (119) in the text). It can happen that these solutions sum up to zero in certain spin-sectors. However, in section 6, we will explicitly calculate the eigenvalues of these solutions for the spin-zero sector of the four chains and verify that they are non-vanishing. They yield the correct eigenvalues of the transfer matrix and give the correct Bethe roots.

1.3.1. Infinite-dimensional symmetries of the six-vertex model. The occurrence of parameter-dependent auxiliary matrices in (24) does not pose a problem for solving the eigenvalue problem of the transfer matrix (3). As we will see in explicit examples the dependence on the additional parameters in *p* drops out of the functional relation (24) when it is written in terms of eigenvalues. This is due to the fact that the dependence on *p* enters either through common normalization factors or the complete N'-strings (14).

Since different auxiliary matrices occur in (24), there is a third possibility. Because of the simultaneous shift in the spectral variable z and the points p, single factors in the eigenvalues of the respective auxiliary matrices can cancel on both sides of the functional relation. Again we will see this realized in two concrete examples in section 6.

As the dependence of the auxiliary matrices Q_p on the point p lifts the degeneracies of the transfer matrix, the cancellation of the additional parameters in p when the operators in (24) are diagonalized manifests the infinite-dimensional non-Abelian symmetry of the six-vertex model at roots of unity. The point $p \in \operatorname{Spec} Z$ can be chosen arbitrarily, therefore one may allow for analytic transformations leaving the complex hypersurface Spec Z invariant. This set of transformations is induced on Spec Z by the action of a non-Abelian infinite-dimensional automorphism group G, called the quantum coadjoint action [20, 21] (see definition (50) in section 2). This provides a geometric interpretation of the infinite-dimensional symmetry of the six-vertex model at $q^N = 1$ for all spin-sectors.

1.4. *Previous results on auxiliary matrices in the literature*

In section 1.2, Baxter's result for the six-vertex auxiliary matrix (18) limited to the spin-sectors $S^z = 0$ has already been mentioned. His result applies to all values of q.

1.4.1. Results for $q^N \neq 1$. Away from a root-of-unity auxiliary matrices for the six-vertex model have been investigated in [25, 27] by considering infinite-dimensional representations of the upper triangular Borel subalgebra of $U_q(\widetilde{sl}_2)$. An extension of the expression (18) to all spin-sectors was recently derived in [27]. In order to solve (9) within the framework of quantum group theory, the choice of an infinite-dimensional auxiliary space seems to be necessary. This leads, however, to technical subtleties as one has to introduce the formal power series

$$\mathcal{N}_\infty(q) \propto \sum_{n\in\mathbb{Z}} q^{nS^z} \tag{26}$$

in the normalization 'constant' in (18) [27]. For $q^N \neq 1$, the properties of this series are needed to satisfy the functional equation (9) in the sectors $S^z \neq 0$, while $\mathcal{N}_\infty$ has to be removed in the sectors $S^z = 0$ where it becomes ill-defined. When the root-of-unity limit is taken, $q^N \to 1$, the expression (18) without the factor (26) solves the functional relation (9) only in the spin-sectors $S^z = 0 \bmod N$ [27].

1.4.2. Results for $q^N = 1$. The Yang–Baxter algebra (19) of the six-vertex model at roots of unity has been the starting point for previous investigations in the literature [24, 26, 35, 36], in particular with regard to the chiral Potts model [37, 38]. For a certain choice of a root-of-unity representation, the intertwiner (22) is contained in the solutions discussed in [24, 26, 36]. (The relation will be described at the end of section 5 in this paper.) However, in this representation, one cannot take the limit from cyclic to nilpotent representations (see section 2 for an explanation of these terms). This limit is needed for discussing even roots of unity which have been excluded in [24, 26, 36].

As explained above, one of the main results in this work is the connection with representation theory and the results of [20, 21]. For this, the solution of (22) for any root-of-unity representation is needed which is given in section 3 of this work (cf (68) and (71)).

Moreover, in [24, 26], five parameter families of auxiliary matrices at roots of unity are introduced. In contrast to the construction in this paper, the authors of [24, 26] discuss solutions to the functional equation (9) in Baxter's approach.

The equation (24) and the exact sequence (25) are new results. Again the precise relation between the outcome of [24, 26] and the results presented here is explained at the end of section 5.

In [35], the connection between representations of $U_q(\widetilde{sl}_{2,3})$ and the chiral Potts model has been considered. The construction of auxiliary matrices and the decomposition of the tensor product via (25) have not been analysed.

1.5. *Outline of the paper*

In section 2, the representation theoretic results on the quantum groups $U_q(\widetilde{sl}_2)$ and $U_q(sl_2)$, we shall need, are briefly summarized. In particular, the enlarged centre of the algebras, the concept of evaluation representations and the existence criteria for intertwiners are reviewed. Also the hypersurface whose points will label the auxiliary matrices and the infinite-dimensional automorphism group G are introduced.

Section 3 gives a concrete solution for the intertwiner (22) which is the basic constituent for the construction of the auxiliary matrices. Its transformation properties under spin-reversal and the difference between principal and homogeneous gradation are discussed.

In section 4, the exact sequence (25) is described. In particular, it is stated in terms of representation theory how the points p', p'' are related to p.

Section 5 contains the definition of the auxiliary matrices and the proof of the functional equation (24). In addition, the transformation properties of the auxiliary matrices under the finite symmetries (6), (7) and the commutation relations of the operators in (24) are discussed.

In section 6, the construction procedure of the auxiliary matrix is illustrated for the two simple examples $N = 3$, $M = 3, 4$. The eigenvalues of the transfer matrix are calculated from the ones of the auxiliary matrix and the Bethe roots among the zeros of the auxiliary matrix are identified. As expected, they satisfy the Bethe ansatz equations (2). One example of a complete N-string is given and it is shown that its centre is determined via the central elements of the quantum group. Furthermore, we will see that the auxiliary matrices constructed in this paper do not coincide with the expression (18).

Section 7 summarizes the results and gives the conclusions. It is also explained how the auxiliary matrices can be equipped with the quantum coadjoint action.

2. Quantum groups at roots of unity—a reminder

In this section, the known results about representations of $U_q(\widetilde{sl}_2)$, $U_q(sl_2)$ at a root of unity are briefly reviewed focussing only on those facts which are necessary for our discussion. The original references for the results stated are [20–22]. The material may also be found in several textbooks [39, 40].

2.1. *The finite quantum group* $U_q(sl_2)$

For simplicity, let us start with the finite quantum group $U_q(sl_2)$ defined in terms of the Chevalley generators $\{e, f, k\}$ obeying the algebraic relations

$$kek^{-1} = q^2 e \qquad kfk^{-1} = q^{-2} f \qquad [e, f] = \frac{k - k^{-1}}{q - q^{-1}}. \tag{27}$$

There exists a unique Hopf algebra structure on $U_q(sl_2)$ with comultiplication Δ, counit ε and antipode Γ such that

$$\begin{array}{lll} \Delta(e) = e \otimes 1 + k \otimes e & \Delta(f) = f \otimes k^{-1} + 1 \otimes f & \Delta(k) = k \otimes k \\ \Gamma(k) = k^{-1} & \Gamma(e) = -k^{-1} e & \Gamma(f) = -fk \\ \varepsilon(e) = \varepsilon(f) = 0 & \varepsilon(k) = 1. & \end{array} \tag{28}$$

The opposite coproduct Δ^{op} is obtained by permuting the two factors. There is an alternative variant of this quantum algebra which occurs in the literature and will be important for our discussion. Suppose that $q^4 \neq 0, 1$. Then we denote by $\check{U}_q(sl_2)$ the algebra generated by $\{\check{e}, \check{f}, t\}$ subject to the relations

$$t\check{e}t^{-1} = q\check{e} \qquad t\check{f}t^{-1} = q^{-1}\check{f} \qquad [\check{e}, \check{f}] = \frac{t^2 - t^{-2}}{q - q^{-1}}. \tag{29}$$

The Hopf algebra structure is now given by

$$\begin{array}{lll} \Delta(\check{e}) = \check{e} \otimes t^{-1} + t \otimes \check{e} & \Delta(\check{f}) = \check{f} \otimes t^{-1} + t \otimes \check{f} & \Delta(t) = t \otimes t \\ \Gamma(\check{e}) = -q\check{e} & \Gamma(\check{f}) = -q^{-1}\check{f} & \Gamma(t) = t^{-1} \\ \varepsilon(\check{e}) = \varepsilon(\check{f}) = 0 & \varepsilon(t) = 1. & \end{array} \tag{30}$$

The algebras $U_q(sl_2)$ and $\breve{U}_q(sl_2)$ are not isomorphic. But there exists an injective Hopf algebra homomorphism $U_q(sl_2) \hookrightarrow \breve{U}_q(sl_2)$ via the embedding

$$e \hookrightarrow \breve{e}t \qquad f \hookrightarrow t^{-1}\breve{f} \qquad k \hookrightarrow t^2. \tag{31}$$

Thus, we can view $U_q(sl_2)$ as an Hopf subalgebra of $\breve{U}_q(sl_2)$. For the moment, we concentrate on $U_q(sl_2)$ but we will return to the algebra $\breve{U}_q(sl_2)$ when discussing the intertwiner (22).

For values of the deformation parameter q different from a root of unity, the centre of the algebra $U_q(sl_2)$ is generated by the Casimir element,

$$\mathbf{c} = qk + q^{-1}k^{-1} + (q - q^{-1})^2 fe. \tag{32}$$

Henceforth let $q^N = 1$, $N \geqslant 3$ with q being primitive. Then the centre of the quantum group is enlarged by the additional central elements,

$$\mathbf{x} = ((q-q^{-1})e)^{N'} \qquad \mathbf{y} = ((q-q^{-1})f)^{N'} \qquad \mathbf{z}^{\pm 1} = k^{\pm N'} \qquad N' = \begin{cases} N & N \text{ odd} \\ N/2 & N \text{ even.} \end{cases} \tag{33}$$

The presence of additional central elements at $q^N = 1$ compared with $q^N \neq 1$ considerably enriches the representation theory and will reflect on an algebraic level the additional symmetry encountered in the six-vertex model.

In the following, we denote by Z_0 the commutative subalgebra generated by the elements (33) and $Z = Z_0 \cup \{\mathbf{c}\}$ the centre of $U_q(sl_2)$. It is important to note that $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are algebraically independent, while the Casimir element is algebraic over Z_0. To express this algebraic dependence, we define

$$F_N(x) = \begin{cases} \prod_{\ell=0}^{N-1}(x + q^\ell + q^{-\ell}) - 2 & N \text{ odd} \\ \prod_{\ell=0,\text{even}}^{N-1}(x - q^{\ell+1} - q^{-\ell-1}) - 2 & N \text{ even.} \end{cases} \tag{34}$$

Then the algebraic relation between the elements (33) and the Casimir operator is given by [20],

$$\mathbf{xy} + (-1)^{N+1}(\mathbf{z} + \mathbf{z}^{-1}) = F_N(\mathbf{c}). \tag{35}$$

According to Schur's lemma, the elements in the centre act as scalars in any finite-dimensional irreducible representation π. They can be therefore treated as ordinary complex numbers and by abuse of notation $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{c}$ will sometimes stand for the algebraic elements (33), (32) and sometimes for their numerical values in some (unspecified) representation. Denote by Rep $U_q(sl_2)$, the set of equivalence classes $[\pi]$ of finite-dimensional irreducible representations π and define the following hypersurface in $\mathbb{C}^4$:

$$\text{Spec}\, Z = \{p = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{c}) | \mathbf{xy} + (-1)^{N+1}(\mathbf{z} + \mathbf{z}^{-1}) = F_N(\mathbf{c})\}. \tag{36}$$

Moreover, there exists the following sequence of surjective maps [20],

$$\text{Rep}\, U_q(sl_2) \xrightarrow{\mathfrak{X}} \text{Spec}\, Z \to \text{Spec}\, Z_0 = \mathbb{C}^2 \times \mathbb{C}^\times \tag{37}$$

where the map $\text{Rep}\, U_q(sl_2) \to \text{Spec}\, Z$ assigns to each equivalence class its values of the central elements,

$$\mathfrak{X} : [\pi] \to p = (\pi(\mathbf{x}), \pi(\mathbf{y}), \pi(\mathbf{z}), \pi(\mathbf{c})). \tag{38}$$

Away from the singular points

$$D = \begin{cases} \{(0, 0, \pm 1, \pm(q^\ell + q^{-\ell})) | 1 \leqslant \ell \leqslant N-1\} & N \text{ odd} \\ \{(0, 0, (-1)^{\ell-1}, (q^\ell + q^{-\ell})) | 1 \leqslant \ell \leqslant N-1, \ell \neq N'\} & N \text{ even.} \end{cases} \tag{39}$$

$\mathfrak{X}$ is not only surjective but also injective. In other words, the values of the central elements $(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{c})$ fix the representation up to isomorphism provided $\mathfrak{X}[\pi] \notin D$. The projection Spec $Z \to$ Spec Z_0,

$$p = (\pi(\mathbf{x}), \pi(\mathbf{y}), \pi(\mathbf{z}), \pi(\mathbf{c})) \to p_* = (\pi(\mathbf{x}), \pi(\mathbf{y}), \pi(\mathbf{z})) \tag{40}$$

however, is not injective, its inverse image consists in general of N' points. Away from the discriminant set D the fiber in Spec Z over the base point $p_* = (\mathbf{x}, \mathbf{y}, \mathbf{z})$ is given by the set

$$p \notin D : \quad \{p_\ell = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mu q^\ell + \mu^{-1} q^{-\ell}) | \ell \in \mathbb{Z}_{N'}\}. \tag{41}$$

(The parameter μ has been implicitly defined in (23).) This is immediate to see when employing the identity

$$\mathbf{xy} + (-1)^{N+1}(\mathbf{z} + \mathbf{z}^{-1}) = F_N(\mu + \mu^{-1}) = \mu^{N'} + \mu^{-N'}. \tag{42}$$

There is a fundamental difference between the irreducible representations whose image under $\mathfrak{X}$ lies in the discriminant set (39) and those for which it does not. Let $\mathfrak{X}[\pi] \in D$ then the representation π can be obtained by taking the limit $q^N \to 1$ of some standard representations at $q^N \neq 1$ with dimension $\dim \pi < N'$. These representations are the quantum analogue of representations of the non-deformed algebra sl_2.

Those representations whose image $\mathfrak{X}[\pi]$ lies outside the discriminant set (39) are N'-dimensional and have no 'classical' counterparts. They depend in general on three parameters, $\pi = \pi_N^{\xi,\zeta,\lambda}$ with $\xi, \zeta, \lambda \in \mathbb{C}^2 \times \mathbb{C}^\times$. Let $v_n, n = 0, 1, \ldots, N' - 1$ denote the canonical basis in $\mathbb{C}^{N'}$ then the representation $\pi_N^{\xi,\zeta,\lambda}$ is defined via the relations [20, 41]

$$\begin{aligned}
&\pi_N^{\xi,\zeta,\lambda}(k)v_n = \lambda q^{-2n} v_n \\
&\pi_N^{\xi,\zeta,\lambda}(f)v_n = v_{n+1} \\
&\pi_N^{\xi,\zeta,\lambda}(f)v_{N'-1} = \zeta v_0 \\
&\pi_N^{\xi,\zeta,\lambda}(e)v_n = ([\lambda; n-1]_q [n]_q + \xi\zeta) v_{n-1} \qquad n > 0 \\
&\pi_N^{\xi,\zeta,\lambda}(e)v_0 = \xi v_{N'-1}
\end{aligned} \tag{43}$$

with

$$[\lambda; n]_q := \frac{\lambda q^{-n} - \lambda^{-1} q^n}{q - q^{-1}} \qquad [n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{44}$$

The values of the central elements (33) in terms of the parameter set (ξ, ζ, λ) are given by

$$\begin{aligned}
&\pi_N^{\xi,\zeta,\lambda}(\mathbf{x}) = \xi (q - q^{-1})^{N'} \prod_{n=1}^{N'-1} ([\lambda; n-1]_q [n]_q + \xi\zeta) =: (q - q^{-1})^{N'} \eta \\
&\pi_N^{\xi,\zeta,\lambda}(\mathbf{y}) = \zeta (q - q^{-1})^{N'} \qquad \pi_N^{\xi,\zeta,\lambda}(\mathbf{z}) = \lambda^{N'}
\end{aligned} \tag{45}$$

and

$$\pi_N^{\xi,\zeta,\lambda}(\mathbf{c}) = q\lambda + q^{-1}\lambda^{-1} + (q - q^{-1})^2 \xi\zeta. \tag{46}$$

Note that the representations (43) provide coordinates on the hypersurface (36) via (45) and (46),

$$\varphi : \mathbb{C} \times \mathbb{C} \times \mathbb{C}^\times \to \text{Spec } Z \backslash D, (\xi, \zeta, \lambda) \to \varphi(\xi, \zeta, \lambda) = \mathfrak{X}\big[\pi_N^{\xi,\zeta,\lambda}\big]. \tag{47}$$

In the subsequent sections, this coordinate map will be often used to perform calculations in a concrete representation. However, the results obtained will not depend on this particular choice of 'coordinates'.

Throughout this paper, it will be important to distinguish between certain subvarieties of representations in Spec Z_0, Spec Z. We will use the nomenclature presented in the table below.

Table 1. The different types of representations of $U_q(sl_2)$ at a root of unity.

Representation	$\pi(\mathbf{x})$	$\pi(\mathbf{y})$	$\pi(\mathbf{z})$
Nilpotent	0	0	$\mathbb{C}^\times$
Semi-cyclic	$0(\mathbb{C}^\times)$	$\mathbb{C}^\times(0)$	$\mathbb{C}^\times$
Cyclic	$\mathbb{C}^\times$	$\mathbb{C}^\times$	$\mathbb{C}^\times$

Note that nilpotent and semi-cyclic representations possess a highest or lowest weight vector, while cyclic representations do not. This can be explicitly seen in the concrete representation (43). Applying the generator e or f to a basis vector N' times yields the same vector again, hence the name cyclic representation.

2.2. *The quantum coadjoint action*

Following [20, 21], one can define for $q^N \neq 1$ the following infinitesimal automorphisms on $U_q(sl_2)$:

$$\underset{\sim}{e}(x) = [e^{N'}/[N']_q!, x] \qquad \underset{\sim}{f}(x) = [f^{N'}/[N']_q!, x] \qquad \underset{\sim}{k}(x) = [k^{N'}/[N']_q!, x] \tag{48}$$

with $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$. Noteworthy, the above derivations stay well defined in the root of unity limit $q^N \to 1$ where their action on the Chevalley–Serre generators reads

$$\begin{aligned} &\underset{\sim}{e}(e) = 0 \qquad \underset{\sim}{e}(f) = \frac{kq - k^{-1}q^{-1}}{q - q^{-1}} \frac{e^{N-1}}{[N-1]!} \qquad \underset{\sim}{e}(k^{\pm 1}) = \mp N^{-1}\mathbf{x}k^{\pm 1} \\ &\underset{\sim}{f}(e) = -\frac{f^{N-1}}{[N-1]!}\frac{kq - k^{-1}q^{-1}}{q - q^{-1}} \qquad \underset{\sim}{f}(f) = 0 \qquad \underset{\sim}{f}(k^{\pm 1}) = \pm N^{-1}\mathbf{y}k^{\pm 1}. \end{aligned} \tag{49}$$

Of particular interest is their action on the central elements (33),

$$\begin{aligned} &\underset{\sim}{e}(\mathbf{x}) = 0 \qquad \underset{\sim}{e}(\mathbf{y}) = \mathbf{z} - \mathbf{z}^{-1} \qquad \underset{\sim}{e}(\mathbf{z}^{\pm 1}) = \mp\mathbf{x}\mathbf{z}^{\pm 1} \\ &\underset{\sim}{f}(\mathbf{y}) = 0 \qquad \underset{\sim}{f}(\mathbf{x}) = \mathbf{z}^{-1} - \mathbf{z} \qquad \underset{\sim}{f}(\mathbf{z}^{\pm 1}) = \pm\mathbf{y}\mathbf{z}^{\pm 1} \qquad \underset{\sim}{k}(\mathbf{x}) = \mathbf{x}\mathbf{z} \end{aligned}$$

from which one deduces that their exponentials yield analytic transformations on the hypersurface (36),

$$\begin{aligned} &\exp(t\underset{\sim}{e})\mathbf{x} = \mathbf{x} \qquad \exp(t\underset{\sim}{e})\mathbf{z}^{\pm 1} = \mathrm{e}^{\mp t\mathbf{x}}\mathbf{z}^{\pm 1} \qquad \exp(t\underset{\sim}{e})\mathbf{y} = \mathbf{y} - \left(\mathbf{z}\,\frac{\mathrm{e}^{-t\mathbf{x}} - 1}{\mathbf{x}} + \mathbf{z}^{-1}\,\frac{\mathrm{e}^{t\mathbf{x}} - 1}{\mathbf{x}}\right) \\ &\exp(t\underset{\sim}{f})\mathbf{y} = \mathbf{y} \qquad \exp(t\underset{\sim}{f})\mathbf{z}^{\pm 1} = \mathrm{e}^{\pm t\mathbf{y}}\mathbf{z}^{\pm 1} \qquad \exp(t\underset{\sim}{f})\mathbf{x} = \mathbf{x} + \left(\mathbf{z}\,\frac{\mathrm{e}^{-t\mathbf{y}} - 1}{\mathbf{y}} + \mathbf{z}^{-1}\,\frac{\mathrm{e}^{t\mathbf{y}} - 1}{\mathbf{y}}\right). \end{aligned} \tag{50}$$

Here $t \in \mathbb{C}$ is a free parameter. The group G generated by these automorphisms is infinite-dimensional and its action on the hypersurface is called the quantum coadjoint action [20, 21]. The fixed point set under this action is given by the discriminant set (39). Note further that the Casimir element remains invariant, i.e., the polynomial $\mathbf{xy} + \mathbf{z} + \mathbf{z}^{-1}$ is fixed under the action of G.

2.3. The quantum loop algebra $U_q(\widetilde{sl}_2)$

In order to make contact with the six-vertex model (4), one needs to consider instead of the finite quantum group $U_q(sl_2)$ the quantum loop algebra $U_q(\widetilde{sl}_2)$. For its definition, we assume, temporarily q to be generic. The quantum loop algebra is defined through the algebraic relations

$$k_i e_j k_i^{-1} = q^{A_{ij}} e_j \qquad k_i f_j k_i^{-1} = q^{-A_{ij}} f_j \qquad k_i k_j = k_j k_i \qquad i, j = 0, 1 \tag{51}$$

where the Cartan matrix A is

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

In addition one has to impose the Chevalley–Serre relations,

$$\begin{aligned} & e_i^3 e_j - [3]_q e_i^2 e_j e_i + [3]_q e_i e_j e_i^2 - e_j e_i^3 = 0 \\ & f_i^3 f_j - [3]_q f_i^2 f_j f_i + [3]_q f_i f_j f_i^2 - f_j f_i^3 = 0 \qquad i \neq j \quad i, j = 0, 1. \end{aligned} \tag{52}$$

Similar to the case $U_q(sl_2)$, the quantum loop algebra $U_q(\widetilde{sl}_2)$ can be made into a Hopf algebra using the definitions analogous to (28). Again we may view $U_q(\widetilde{sl}_2)$ as a subalgebra of the larger Hopf algebra $\breve{U}_q(\widetilde{sl}_2)$ which is the counterpart to $\breve{U}_q(sl_2)$. The quantum loop algebra at roots of unity is then obtained via the specialization map (see 1.9 on p 398 in [22]).

A complete classification of the irreducible representations of the non-restricted algebra $U_q(\widetilde{sl}_2)$ at roots of unity is presently not known [22]. In contrast to the $U_q(sl_2)$, the quantum loop algebra is not finitely generated over its centre $\tilde{Z}$ and the structure of Spec $\tilde{Z}$ is less well understood. In particular, the canonical map Rep $U_q(\widetilde{sl}_2) \to$ Spec $\tilde{Z}$ is neither surjective or bijective in general [22]. It has been demonstrated, however, that all possible values of the central elements can be obtained by considering irreducible subquotients of tensor products of evaluation representations [22].

An evaluation representation is constructed by composing a representation of the finite quantum group with the evaluation homomorphism $\mathrm{ev}_w : U_q(\widetilde{sl}_2) \to U_q(sl_2)$ defined by [42]

$$\begin{aligned} & e_0 \to w f \qquad f_0 \to w^{-1} e \qquad k_0 \to k^{-1} \\ & e_1 \to e \qquad f_1 \to f \qquad k_1 \to k \qquad w \in \mathbb{C}. \end{aligned} \tag{53}$$

In the following, I shall restrict discussion to such evaluation representations and define

$$\pi_w^p \equiv \pi^p \circ \mathrm{ev}_w \qquad p \in \mathrm{Spec}\, Z \backslash D \qquad [\pi^p] = \mathfrak{X}^{-1}(p). \tag{54}$$

The choice to consider only a single evaluation representation will keep the subsequent calculations feasible. Obviously, the representation π^p is only determined up to isomorphism. For some calculations, it will be necessary to remove this ambiguity. This can be achieved by employing the coordinate map (47),

$$\pi^p \equiv \pi_{N'}^{\xi,\zeta,\lambda} \qquad (\xi, \zeta, \lambda) = \varphi^{-1}(p) \tag{55}$$

which gives now a concrete realization according to (43). From the definition (54), one deduces the six central values

$$\pi_w^p(\mathbf{x}_0)/w^{N'} = \pi_w^p(\mathbf{y}_1) = (q - q^{-1})^{N'} \eta \tag{56}$$

$$\pi_w^p(\mathbf{y}_0) w^{N'} = \pi_w^p(\mathbf{x}_1) = (q - q^{-1})^{N'} \zeta \qquad \pi_w^p\big(\mathbf{z}_0^{\mp 1}\big) = \pi_w^p\big(\mathbf{z}_1^{\pm 1}\big) = \lambda^{\pm N'} \tag{57}$$

leaving only the four free parameters w and $\varphi^{-1}(p) = (\xi, \zeta, \lambda)$.

2.4. Necessary existence criteria for intertwiners

Besides the occurrence of irreducible representations depending on continuous parameters, there is another important characteristic feature of quantum groups at roots of unity. The quantum group $U_q(\widetilde{sl}_2)$ at a root of unity is no longer quasi-triangular and the concept of a universal *R*-matrix present at $q^N \neq 1$ is problematic. Now one has to satisfy certain necessary existence criteria for intertwiners. The argument is by now standard (see, e.g., [39, 40]), and it is worthwhile to repeat it in order to show why one can find an intertwiner (22) for generic representations when q is an odd primitive root of unity but not for an even one. Taking into account that the central subalgebra (33) forms a Hopf subalgebra,

$$\Delta(\mathbf{x}_i) = (e_i \otimes 1 + k_i \otimes e_i)^{N'} = \mathbf{x}_i \otimes 1 + \mathbf{z}_i \otimes \mathbf{x}_i$$
$$\Delta(\mathbf{y}_i) = \left(f_i \otimes k_i^{-1} + 1 \otimes f_i\right)^{N'} = \mathbf{y}_i \otimes \mathbf{z}_i^{-1} + 1 \otimes \mathbf{y}_i$$
$$\Delta(\mathbf{z}_i) = \mathbf{z}_i \otimes \mathbf{z}_i$$

an intertwiner between two representations π, π' at $q^N = 1$ can only exist if the following equalities hold:

$$\begin{aligned} &\pi(\mathbf{x}_i) + \pi(\mathbf{z}_i)\pi'(\mathbf{x}_i) = \pi'(\mathbf{x}_i) + \pi(\mathbf{x}_i)\pi'(\mathbf{z}_i) \\ &\pi(\mathbf{y}_i)\pi'\left(\mathbf{z}_i^{-1}\right) + \pi'(\mathbf{y}_i) = \pi'(\mathbf{y}_i)\pi\left(\mathbf{z}_i^{-1}\right) + \pi(\mathbf{y}_i). \end{aligned} \tag{58}$$

As the limit $q^N \to 1$ of the six-vertex model is connected with the two-dimensional nilpotent representation (21),

$$q^N \to 1 : \pi_z^0(\mathbf{x}_i) = \pi_z^0(\mathbf{y}_i) = 0 \qquad \pi_z^0(\mathbf{z}_i) = q^{N'} = \pm 1 \qquad i = 0, 1 \tag{59}$$

one arrives at the conditions

$$\pi_w^p(\mathbf{x}_i) = \pi_w^p(\mathbf{x}_i)\pi_z^0(\mathbf{z}_i) \qquad \text{and} \qquad \pi_w^p(\mathbf{y}_i) = \pi_w^p(\mathbf{y}_i)\pi_z^0(\mathbf{z}_i) \tag{60}$$

for an intertwiner to exist in the case of the tensor product $\pi_w^p \otimes \pi_z^0$. If we take N to be odd then $\pi_z^0(\mathbf{z}_i) = 1$ and there are no restrictions on the existence of an intertwiner between π_z^0 and some cyclic, semi-cyclic or nilpotent evaluation representation π_w^p at a root of unity. For N even $\pi_z^0(\mathbf{z}_i) = -1$ and an intertwiner can only exist for nilpotent representations, i.e., $\pi^p(\mathbf{x}) = \pi^p(\mathbf{y}) = 0$.

3. Intertwiner for $\pi_w^p \otimes \pi_z^0$

Having investigated the existence criteria for intertwiners at a root of unity, we now need to explicitly construct the operator (22) which will provide us with the basic constituent of the auxiliary matrix (16). For this purpose, it will be necessary to consider the larger Hopf algebra $\breve{U}_q(\widetilde{sl}_2)$ respectively $\breve{U}_q(sl_2)$ as the matrix elements of the intertwiner will lie in these algebras and not in $U_q(\widetilde{sl}_2), U_q(sl_2)$. This becomes important since there is a crucial difference between the representations of $U_q(sl_2)$ for N odd and N even. Given any point $p = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{c}) \in \operatorname{Spec} Z$ all of the associated representations π^p in the class $\mathfrak{X}^{-1}(p)$ can be extended to a representation of $\breve{U}_q(sl_2)$ when q is an odd root of unity. For example, let $p = \varphi(\xi, \zeta, \lambda)$ with respect to the coordinate map (47). Then setting

$$\pi_N^{\xi,\zeta,\lambda}(t) := \pi_N^{\xi,\zeta,\lambda}(k)^{\frac{1}{2}} \qquad \text{with} \quad \pi_N^{\xi,\zeta,\lambda}(k)^{\frac{1}{2}} v_n = \lambda^{\frac{1}{2}} q^{-n} v_n \tag{61}$$

and

$$\pi_N^{\xi,\zeta,\lambda}(\breve{e}) := \pi_N^{\xi,\zeta,\lambda}(e)\pi_N^{\xi,\zeta,\lambda}(k)^{-\frac{1}{2}} \qquad \pi_N^{\xi,\zeta,\lambda}(\breve{f}) := \pi_N^{\xi,\zeta,\lambda}(k)^{\frac{1}{2}}\pi_N^{\xi,\zeta,\lambda}(f) \tag{62}$$

gives a well-defined representation of $\breve{U}_q(sl_2)$. For even roots of unity this ceases to be valid, unless the representation is nilpotent, i.e., $\mathbf{x} = \mathbf{y} = 0$. The quantum group relations (29) are not satisfied for cyclic or semi-cyclic representations because of the identities

$$\begin{aligned} \pi_N^{\xi,\zeta,\lambda}(k)^{\frac{1}{2}}\pi_N^{\xi,\zeta,\lambda}(e)\pi_N^{\xi,\zeta,\lambda}(k)^{-\frac{1}{2}}v_0 &= q^{N'+1}\pi_N^{\xi,\zeta,\lambda}(e)v_0 \\ \pi_N^{\xi,\zeta,\lambda}(k)^{\frac{1}{2}}\pi_N^{\xi,\zeta,\lambda}(f)\pi_N^{\xi,\zeta,\lambda}(k)^{-\frac{1}{2}}v_{N'-1} &= q^{N'-1}\pi_N^{\xi,\zeta,\lambda}(f)v_{N'-1}. \end{aligned} \tag{63}$$

This is directly related to the fact that an intertwiner only exists for nilpotent representations π^p as the cyclicity of the representation enforces the above sign change.

In order to unburden the notation let us write temporarily e_i, f_i, k_i for the $N' \times N'$ matrices $\pi_w^p(e_i), \pi_w^p(f_i), \pi_w^p(k_i)$ of $U_q(\widetilde{sl}_2)$ and it will be understood that $\mathbf{x} = \mathbf{y} = 0$ for N even. Recall the defining property of the intertwiner,

$$L^p(w/z)\big(\pi_w^p \otimes \pi_z^0\big)\Delta(x) = \big[\big(\pi_w^p \otimes \pi_z^0\big)\Delta^{\rm op}(x)\big]L^p(w/z) \qquad x \in U_q(\widetilde{sl}_2).$$

To solve this set of equations, it is helpful to decompose the L-matrix over the second factor as (for the moment let us drop the explicit dependence on p in the notation)

$$L = A \otimes \sigma^+\sigma^- + B \otimes \sigma^+ + C \otimes \sigma^- + D \otimes \sigma^-\sigma^+ \qquad A, B, C, D \in \mathrm{End}(\mathbb{C}^{N'}) \tag{64}$$

and to introduce the q-deformed commutator

$$[X, Y]_q = XY - qYX.$$

One then deduces for the Chevalley generators the following commutation relations for the L-matrix entries A, B, C, D,

$$[D, k_i] = 0, [A, k_i] = 0 \qquad k_i B k_i^{-1} = q^{(-)^i 2}B \qquad k_i C k_i^{-1} = q^{(-)^{i+1}2}C \tag{65}$$

$$\begin{aligned} B &= [D, e_0]_q = -k_1[f_1, D]_q = -[A, e_0]_{q^{-1}}k_0^{-1} = [f_1, A]_{q^{-1}} \\ C &= [f_0, D]_{q^{-1}} = -[D, e_1]_{q^{-1}}k_1^{-1} = -k_0[f_0, A]_q = [A, e_1]_q \end{aligned} \tag{66}$$

$$\begin{aligned} &[e_1, C]_q = [f_1, B]_q = [e_0, B]_q = [f_0, C]_q = 0 \\ &[C, e_0]_q = A - Dk_0 \qquad [C, f_1]_q = q\big(k_1^{-1}A - D\big) \\ &[B, e_1]_q = D - Ak_1 \qquad [B, f_0]_q = q\big(k_0^{-1}D - A\big). \end{aligned} \tag{67}$$

Invoking the quantum group relations (27), (51) and (31), (53), (54), (61), (62), one verifies by direct computation that the ansatz

$$\begin{aligned} A_p &= \rho_+\pi^p(t) - \rho_-\pi^p(t)^{-1} \qquad & B_p &= \rho_+(q - q^{-1})\pi^p(\check{f}) \\ C_p &= \rho_-(q - q^{-1})\pi^p(\check{e}) \qquad & D_p &= \rho_+\pi^p(t)^{-1} - \rho_-\pi^p(t) \end{aligned} \tag{68}$$

yields a valid solution provided we fix the ratio of the coefficients to the specific value

$$\rho_+/\rho_- = qw/z \qquad \rho_\pm = \rho_\pm(w/z, q). \tag{69}$$

Here the representation π^p has been introduced again into the notation to display the explicit dependence of the quantum group generators on $p \in \mathrm{Spec}\, Z$. Note that the normalization functions can be chosen independent of p. This will become important when considering the transformation of the intertwiner under the symmetries of the six-vertex model. Because the representation π^p is only fixed up to isomorphism (68) is defined up to the gauge transformations

$$L^p \to (\phi \otimes 1)L^p(\phi^{-1} \otimes 1) \qquad [\phi\pi^p\phi^{-1}] = [\pi^p]. \tag{70}$$

As pointed out earlier, this ambiguity may be removed by applying the coordinate map (55). For the definition of the auxiliary matrix, this ambiguity is unimportant as the trace is taken in (16).

Since the tensor product is indecomposable for generic values of the ratio z/w, one can conclude that the above solution is the only one up to a normalization factor. In addition, it follows [35] that the intertwiner has to satisfy the Yang–Baxter equation (19). This can also be verified by an explicit calculation.

Note that the intertwiner (22) can be expressed solely in terms of $U_q(sl_2)$ for odd roots of unity upon setting alternatively

$$\begin{aligned} A_p &= \rho_+\pi^p(k)^{\frac{N+1}{2}} - \rho_-\pi^p(k)^{\frac{N-1}{2}} \qquad & B_p &= \rho_+(q-q^{-1})\pi^p(k)^{\frac{N+1}{2}}\pi^p(f) \\ C_p &= \rho_-(q-q^{-1})\pi^p(e)\pi^p(k)^{\frac{N-1}{2}} \qquad & D_p &= \rho_+\pi^p(k)^{\frac{N-1}{2}} - \rho_-\pi^p(k)^{\frac{N+1}{2}}. \end{aligned} \tag{71}$$

The normalization functions $\rho_\pm$ have again to satisfy (69) and can be chosen independent of the parameters p as before. The advantage of this solution is that we can now employ the results on irreducible representations of $U_q(sl_2)$ at roots of unity [20, 21] to the intertwiner. This will allow us to identify auxiliary matrices with points on the hypersurface (36) and to obtain a geometric picture for the symmetries of the six-vertex model.

For a particular root-of-unity representation, the intertwiner (68) is contained in the solutions to the Yang–Baxter algebra obtained in [24, 36]. The relation will be explained at the end of section 5. It is important to note, however, that the expression (68) derived here gives the intertwiner in terms of an arbitrary representation π^p and that only the quantum group relations have to be used in order to verify (22). This is important for two reasons. First, the particular representation used in [24] is not suitable for discussing the nilpotent limit which has to be taken when N is even. Second, the expression (71) displaying the dependence on the quantum group generators will ultimately allow us to identify equivalent auxiliary matrices by associating them with points in the hypersurface (36).

3.1. Transformation under spin-reversal

As the six-vertex model is invariant under spin-reversal, we need to investigate the behaviour of the constructed intertwiner under this transformation. From the decomposition (64), one deduces the simple transformation property

$$(1\otimes\sigma^x)L(1\otimes\sigma^x) = \begin{pmatrix} D & C \\ B & A \end{pmatrix}. \tag{72}$$

This transformation can be interpreted in terms of representation theory by noting that the algebraic relations resulting from the intertwiner condition stay invariant provided we apply simultaneously the $U_q(\widetilde{sl}_2)$ algebra automorphism $(e_i, f_i, k_i) \xrightarrow{\hat\omega} (f_{i+1}, e_{i+1}, k_{i+1}), i \in \mathbb{Z}_2$. Hence, spin-reversal amounts to the replacement

$$\pi_w^p \to \pi^p \circ \mathrm{ev}_w \circ \hat\omega$$

in the intertwiner equation (22). In order to obtain again an evaluation representation of the form (54) it is of advantage to rewrite this in terms of the $U_q(sl_2)$ algebra automorphism given by

$$\omega(e) = f \qquad \omega(f) = e \qquad \omega(k) = k^{-1}. \tag{73}$$

Introducing on the hypersurface (36) the map

$$\mathrm{Spec}\, Z \ni p = (\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{c}) \to \Re p := (\mathbf{y},\mathbf{x},\mathbf{z}^{-1},\mathbf{c}) \tag{74}$$

and observing that the value of the Casimir element (32) stays invariant under the application of ω,

$$\mathbf{c} = qk + q^{-1}k^{-1} + (q - q^{-1})^2 fe = qk^{-1} + q^{-1}k + (q - q^{-1})^2 ef \tag{75}$$

one immediately verifies the following equality of equivalence classes:

$$\left[\pi_w^{\mathfrak{R}p}\right] = [\pi^p \circ \omega \circ \mathrm{ev}_w]. \tag{76}$$

Hence, there exists a non-singular $N' \times N'$ matrix ϕ transforming one representation into the other and an elementary calculation now shows that for the intertwiner solution (71) one has

$$\begin{aligned}
A_p &= \mathbf{z}\big(\rho_+ \pi^p(k)^{-\frac{N-1}{2}} - \rho_- \pi^p(k)^{-\frac{N+1}{2}}\big) = \mathbf{z}\phi D_{\mathfrak{R}p}\phi^{-1} \\
B_p &= \mathbf{z}\rho_- w(q - q^{-1})\pi^p(f)\pi^p(k)^{-\frac{N-1}{2}} = \mathbf{z}w\phi C_{\mathfrak{R}p}\phi^{-1} \\
C_p &= \mathbf{z}\rho_+ w^{-1}(q - q^{-1})\pi^p(k)^{-\frac{N+1}{2}} e = \mathbf{z}w^{-1}\phi B_{\mathfrak{R}p}\phi^{-1} \\
D_p &= \mathbf{z}\big(\rho_+ \pi^p(k)^{-\frac{N+1}{2}} - \rho_- \pi^p(k)^{-\frac{N-1}{2}}\big) = \mathbf{z}\phi A_{\mathfrak{R}p}\phi^{-1}.
\end{aligned} \tag{77}$$

For the alternative solution (68), whose matrix elements lie in $\breve{U}_q(sl_2)$, one obtains an analogous result with the exception that the factor $\mathbf{z}$ needs to be omitted. In summary, we therefore obtain the transformation law,

$$(1 \otimes \sigma^x)L^p(w)(1 \otimes \sigma^x) = \mathbf{z}\big(\phi \otimes w^{-\frac{\sigma^z}{2}}\big)L^{\mathfrak{R}p}(w)\big(\phi^{-1} \otimes w^{\frac{\sigma^z}{2}}\big). \tag{78}$$

Again the factor $\mathbf{z}$ is absent for the other solution (68). In terms of the coordinate map (47) spin-reversal then amounts to the coordinate change

$$\varphi^{-1}(p) = (\xi, \zeta, \lambda) \to (\xi^{\mathfrak{R}}, \eta, \lambda^{-1}q^{-2}) = \varphi^{-1}(\mathfrak{R}p) \tag{79}$$

where η is given by (45) and the parameter $\xi^{\mathfrak{R}}$, which depends on (η, ζ, λ), is chosen such that

$$\zeta = \xi^{\mathfrak{R}} \prod_{n=0}^{N'-1} ([\lambda^{-1}q^{-2}; n-1][n] + \xi^{\mathfrak{R}}\eta). \tag{80}$$

On basis of this result, we will derive the transformation property of the auxiliary matrix under spin-reversal. The additional spectral parameter dependence in (78) induced by the spin-reversal transformation can be avoided when changing to the principal gradation.

3.2. *The principal gradation*

When defining the evaluation homomorphism $U_q(\widetilde{sl}_2) \to U_q(sl_2)$, one has the choice between two gradations of the loop algebra. The definition (53) corresponds to the homogeneous gradation and is associated with the degree operator

$$[d, e_i] = \delta_{i0}e_i \qquad [d, f_i] = -\delta_{i0}f_i \qquad [d, k_i] = 0$$

while the principal gradation is induced by

$$[\hat{\rho}, e_i] = e_i \qquad [\hat{\rho}, f_i] = -f_i \qquad [\hat{\rho}, k_i] = 0.$$

Both degrees are related by the identity $\hat{\rho} = h^\vee d + \rho$ with $h^\vee = 2$ being the dual Coxeter number and ρ the Weyl vector. While the homogeneous gradation will mainly be used throughout this paper for algebraic simplicity, the principal gradation is more natural in order to discuss the behaviour of the intertwiner (22) under spin-reversal. The evaluation homomorphism then reads

$$\begin{aligned}
&e_0 \to xf \qquad f_0 \to x^{-1}e \qquad k_0 \to k^{-1} \\
&e_1 \to xe \qquad f_1 \to x^{-1}f \qquad k_1 \to k \qquad x = z^{\frac{1}{2}} \in \mathbb{C}.
\end{aligned} \tag{81}$$

This change from the homogeneous gradation to the principal one is reflected by the following well-known gauge transformation of the six-vertex R -matrix (4),

$$\mathcal{R}(x) = \big(x^{\frac{\sigma^z}{2}} \otimes 1\big) R(x^2)\big(x^{-\frac{\sigma^z}{2}} \otimes 1\big) = \big(1 \otimes x^{-\frac{\sigma^z}{2}}\big) R(x^2)\big(1 \otimes x^{\frac{\sigma^z}{2}}\big). \tag{82}$$

The corresponding Boltzmann weights read in this gauge

$$a = \varrho \qquad b = \varrho \frac{(1-x^2)q}{1-x^2q^2} \qquad c = c' = \varrho \frac{(1-q^2)x}{1-x^2q^2} \qquad \varrho(x,q) = \rho(x^2,q). \tag{83}$$

Obviously, the six-vertex R-matrix in the homogenous gauge (4) violates spin-reversal symmetry while the one in the principal gauge (82) is invariant. Clearly, this gauge change does not matter on the level of the transfer matrix (3) (upon identifying $z = x^2$) as the gauge transformation can be invoked in the auxiliary space over which the trace is taken. However, this argument ceases to be valid in the case of the auxiliary matrix (16) when cyclic or semi-cyclic representations enter the definition of the L-matrix. Let us define the intertwiner (22) in the principal gradation as

$$\mathcal{L}^p(y) = \big(1 \otimes y^{-\frac{\sigma^z}{2}}\big) L^p(y^2)\big(1 \otimes y^{\frac{\sigma^z}{2}}\big) = \begin{pmatrix} A_p & B_p y^{-1} \\ C_p y & D_p \end{pmatrix} \tag{84}$$

where the coefficients (69) entering the matrices A, B, C, D now have to obey the relation

$$\varrho_+/\varrho_- = qy^2 \qquad \varrho_\pm(y,q) = \rho_\pm(y^2,q). \tag{85}$$

For nilpotent representations, the gauge transformation may also be cast into the form

$$\mathcal{L}^p(y) = (\Gamma_y \otimes 1) L^p(y^2)\big(\Gamma_y^{-1} \otimes 1\big) \qquad (\Gamma_y)_{mn} := \delta_{mn} y^{-\frac{n}{2}}$$

because of the crucial commutation relations

$$\Gamma_y B \Gamma_y^{-1} = y^{-1} B \qquad \text{and} \qquad \Gamma_y C \Gamma_y^{-1} = yC.$$

This does not hold true for $\mathbf{x} \neq 0$ or $\mathbf{y} \neq 0$ due to the cyclicity of the associated representation unless $y^{N'} = 1$. Thus, there is a genuine difference between the principal and the homogenous gauge on the level of the auxiliary matrix (16) when the associated representation is not nilpotent. Returning to the question of the transformation property under spin-reversal, one now calculates

$$\begin{aligned}(1 \otimes \sigma^x)\mathcal{L}^p(y)(1 \otimes \sigma^x) &= \big(1 \otimes y^{\frac{\sigma^z}{2}}\big)(1 \otimes \sigma^x) L^p(y^2)(1 \otimes \sigma^x)\big(1 \otimes y^{-\frac{\sigma^z}{2}}\big) \\ &= \mathbf{z}(\phi \otimes 1)\mathcal{L}^{\mathfrak{R}p}(y)(\phi^{-1} \otimes 1).\end{aligned}$$

This result shows that the additional spectral parameter dependence in (78) originates in the choice of gradation imposed when defining the evaluation homomorphism (53). For the subsequent sections, let us return to the homogeneous gradation unless stated otherwise.

4. Decomposing the tensor product $\pi_w^p \otimes \pi_z^0$

In this section, the decomposition the tensor product $\pi_w^p \otimes \pi_z^0$ via the exact sequence (25) will be described providing the basis for deriving the functional equation (24). As the calculations are straightforward but quite lengthy only the results are presented here. Throughout this section, the coordinate map (47) is applied and all formulas are to be understood with respect to the convention (55). The notation will be simplified by writing simply $\pi_w, \pi'_{w'}, \pi''_{w''}$ instead of $\pi_w^p, \pi_{w'}^{p'}, \pi_{w''}^{p''}$ with $p = \varphi(\xi, \zeta, \lambda)$, $p' = \varphi(\xi', \zeta', \lambda')$ and $p'' = \varphi(\xi'', \zeta'', \lambda'')$.

Our strategy to determine the exact sequence (25) is as follows: first one considers the action of the quantum loop algebra $U_q(\widetilde{sl}_2)$ on the tensor product $\pi_w \otimes \pi_z^0$. Under the assumption that both of the representations $\pi'_{w'}$ and $\pi''_{w''}$ are of the form (54), this allows us to set up a set of equations for the coefficients of the vectors in the tensor space $\pi_w \otimes \pi_z^0$. They determine the parameters (w, ξ, ζ, λ), $(w', \xi', \zeta', \lambda')$ and $(w'', \xi'', \zeta'', \lambda'')$ labelling the respective representations π_w, $\pi'_{w'}$ and $\pi''_{w''}$. In order to solve the equations for the coefficients, one has to guarantee the vanishing of a determinant. This fixes the value of the ratio $\mu = z/w$ for which the tensor product $\pi_w \otimes \pi_z^0$ is decomposable. One derives that μ is given in terms of the Casimir element (46) through the following quadratic equation:

$$\mu + \mu^{-1} = \xi\zeta(q - q^{-1})^2 + q\lambda + q^{-1}\lambda^{-1} = \pi_N^{\xi,\zeta,\lambda}(\mathbf{c}) \qquad \mu = z/w. \tag{86}$$

The ambiguity in μ is removed by choosing the branch of the square root such that the limit to semi-cyclic or nilpotent representations is consistent (see (95)),

$$\lim_{\xi\to 0} \mu = \lim_{\zeta\to 0} \mu = \lim_{\xi,\zeta\to 0} \mu = \lambda^{-1} q^{-1}. \tag{87}$$

In the following, we regard (z, ξ, ζ, λ) as the independent variables while we have to tune the evaluation parameter w such that μ equals the solution (87) of (86).

4.1. The inclusion $\pi'_{w'} \subset \pi_w \otimes \pi_z^0$

Denote by $\{\uparrow, \downarrow\}$ (spin up, spin down) the standard basis for the two-dimensional evaluation representation (21). The explicit form of the inclusion map defining the subrepresentation $\pi'_{w'}$ in the exact sequence (25) is

$$\iota : \pi'_{w'} \hookrightarrow \pi_w \otimes \pi_z^0 \qquad w'_n \hookrightarrow X_n = \alpha_n v_{n+1} \otimes \uparrow + \beta_n v_n \otimes \downarrow \qquad n \in \mathbb{Z}_{N'}. \tag{88}$$

Here the coefficients are

$$\alpha_n = \zeta^{\delta_{N'-1,n}} q^{-n} \alpha_0 \qquad \text{and} \qquad \beta_n = \frac{\mu q \lambda^{-1} \times q^n - q^{-n}}{q - q^{-1}} \alpha_0. \tag{89}$$

The parameter α_0 is arbitrary unless a specific normalization is chosen. The representation defined by the inclusion (88) is indeed of the form (54) with

$$\pi'_{w'} = \pi_{N'}^{\xi',\zeta',\lambda'} \circ \mathrm{ev}_{w'} \tag{90}$$

and parameters

$$\xi'\zeta' = \xi\zeta \frac{q\mu - \lambda}{\mu - q\lambda} \qquad \zeta' = q^{N'}\zeta \qquad \lambda' = \lambda q^{-1} \qquad w' = wq. \tag{91}$$

The subrepresentation $\pi_{N'}^{\xi',\zeta',\lambda'}$ of the finite quantum group $U_q(sl_2)$ assigns the following values to the central elements (33),

$$\pi_{N'}^{\xi',\zeta',\lambda'}(\mathbf{x}) = (q - q^{-1})^{N'}\eta \qquad \pi_{N'}^{\xi',\zeta',\lambda'}(\mathbf{y}) = (q^2 - 1)^{N'}\zeta \qquad \pi_{N'}^{\xi',\zeta',\lambda'}(\mathbf{z}) = q^{N'}\lambda^{N'} \tag{92}$$

and for the Casimir element, one obtains

$$\pi_{N'}^{\xi',\zeta',\lambda'}(\mathbf{c}) = q\mu + q^{-1}\mu^{-1}. \tag{93}$$

Note that the first identity in (92) is not obvious as one needs to verify the identity

$$\frac{\xi'}{\xi} \prod_{n=1}^{N'-1} \frac{[\lambda'; n-1][n] + \xi'\zeta'}{[\lambda; n-1][n] + \xi\zeta} = \frac{\eta'}{\eta} = 1. \tag{94}$$

From (86) and (93), one infers that the representations $\pi_N^{\xi,\zeta,\lambda}$ and $\pi_N^{\xi',\zeta',\lambda'}$ are in general not isomorphic as they belong to different points in the hypersurface Spec Z. As $q^{N'} = 1$ for N odd, the derived expressions (91) imply, however, that the representations $\pi_{N'}^{\xi,\zeta,\lambda}$ and $\pi_{N'}^{\xi',\zeta',\lambda'}$ correspond to the same point in Spec Z_0. For even roots of unity, this ceases to be valid as $q^{N'} = -1$.

Most of the above formulas stay valid in the limiting case of semi-cyclic (ξ or $\zeta \to 0$) and nilpotent representations ($\xi, \zeta \to 0$). As already mentioned above, the tensor product now becomes reducible at

$$\mu = z/w = 1/q\lambda. \tag{95}$$

The coefficients determining the inclusion map remain unchanged with the only exception that

$$\zeta = 0 \Rightarrow \alpha_{N'-1} = 0. \tag{96}$$

Also, if $\zeta = 0$ and $\xi \neq 0$, the parameters change from (91) to

$$\xi' = \xi q^{N'} \frac{\lambda - \lambda^{-1}}{\lambda q - \lambda^{-1} q^{-1}} \qquad \zeta = \zeta' = 0 \qquad \lambda' = \lambda q^{-1} \tag{97}$$

with the central value η being unchanged,

$$1 = \frac{\xi'}{\xi} \prod_{n=1}^{N'-1} \frac{[\lambda; n][n]}{[\lambda; n-1][n]} = \frac{\eta'}{\eta}. \tag{98}$$

The remaining cases $\xi = 0, \zeta \neq 0$ and $\xi, \zeta = 0$ are obtained by taking the appropriate limit of the previous equations for the cyclic case.

4.2. The quotient representation $\pi''_{w''} = \pi_w \otimes \pi^0_z / \pi'_{w'}$

In order to determine the representation $\pi_w \otimes \pi^0_z / \pi'_{w'}$, one needs to consider the action of the quantum group generators on vectors in the tensor product which do not lie in the representation space $\pi'_{w'}$ and identify the latter with the null space. This is implemented by defining the following projection onto the quotient space via linear extension:

$$\tau : \pi_w \to \pi''_{w''} = \pi_w \otimes \pi^0_z / \pi'_{w'} \qquad \tau(X_n) = 0 \qquad \tau(Y_n) = w''_n \tag{99}$$

with the vectors X_n given by equations (88), (89) and

$$Y_n = \gamma_n v_n \otimes \uparrow, \gamma_n = \prod_{m=1}^{n} \frac{[\lambda''; m-1][m] + \xi''\zeta''}{[\lambda; m-1][m] + \xi\zeta} \gamma_0 \qquad n \in \mathbb{Z}_{N'}. \tag{100}$$

The parameters $(\xi'', \zeta'', \lambda'')$ entering the coefficients γ_n fix the evaluation representation

$$\pi''_{w''} = \pi_N^{\xi'',\zeta'',\lambda''} \circ \mathrm{ev}_{w''} \tag{101}$$

and read explicitly

$$\lambda'' = \lambda q \qquad \xi''\zeta'' = \xi\zeta \frac{\mu q^{-1} - \lambda q^2}{\mu - q\lambda} \qquad \zeta'' = q^{N'}\zeta \qquad w'' = wq^{-1}. \tag{102}$$

Viewing π_w as a representation of the finite quantum group $U_q(sl_2)$, one calculates from these identities the corresponding values of the central elements giving the points in Spec Z_0 and Spec Z. These are

$$\pi_N^{\xi'',\zeta'',\lambda''}(\mathbf{x}) = (q - q^{-1})^{N'}\eta \qquad \pi_N^{\xi'',\zeta'',\lambda''}(\mathbf{y}) = (q^2 - 1)^{N'}\zeta \qquad \pi_N^{\xi'',\zeta'',\lambda''}(\mathbf{z}) = q^{N'}\lambda^{N'} \tag{103}$$

and

$$\pi_N^{\xi'',\zeta'',\lambda''}(\mathbf{c}) = \mu q^{-1} + \mu^{-1} q. \tag{104}$$

Here the identity

$$1 = \eta''/\eta = \xi''/\xi \prod_{n=1}^{N'-1} \frac{[\lambda''; n-1][n] + \xi''\zeta''}{[\lambda; n-1][n] + \xi\zeta} \tag{105}$$

has been employed. For N odd ($q^{N'} = 1$), one now immediately verifies from (103) and (104) that $[\pi_N^{\xi'',\zeta'',\lambda''}]$ shares the same point in the variety Spec Z_0 as $[\pi_N^{\xi,\zeta,\lambda}]$ and $[\pi_N^{\xi',\zeta',\lambda'}]$, but all three representations belong to different points in Spec Z, i.e., they are in general not isomorphic. For even roots of unity ($q^{N'} = -1$), only $[\pi_N^{\xi'',\zeta'',\lambda''}]$ and $[\pi_N^{\xi',\zeta',\lambda'}]$ are mapped onto the same location in Spec Z_0.

There are certain simplifications in the limit of semi-cyclic and nilpotent representations, namely one finds for the coefficients in (100) that

$$\gamma_n = \frac{[\lambda; -1]}{[\lambda; n-1]}\gamma_0.$$

The parameters in the case of semi-cyclic representations are now

$$\xi = 0: \qquad \xi'' = 0 \qquad \zeta'' = q^{N'}\zeta \qquad \lambda'' = \lambda q \qquad w'' = wq^{-1}$$

and

$$\zeta = 0: \qquad \xi'' = \xi\frac{[\lambda; N'-2]}{[\lambda; -1]} \qquad \zeta'' = 0 \qquad \lambda'' = \lambda q \qquad w'' = wq^{-1}.$$

The remaining possibility of nilpotent representations follows from the above by setting $\xi, \zeta = 0$.

5. The *T–Q* functional equation

We are now in the position to derive the functional equation (24) by exploiting the previous results on the decomposition of the tensor product $\pi_w^p \otimes \pi_z^0$ and the explicit construction of the intertwiner (22). Let us start by defining the following family of auxiliary matrices labelled by points on the hypersurface (36):

$$Q_p(z) := \operatorname*{Tr}_{V_0=\pi_w^p} L_{0M}^p(z/\mu_p)\cdots L_{01}^p(z/\mu_p) \qquad \pi_w^p = \pi^p \circ \mathrm{ev}_{w=z/\mu_p}. \tag{106}$$

For odd roots of unity, we employ the solution (71) and for even roots of unity (68). Note that the ambiguity in choosing a representative π^p in the equivalence class $[\pi^p] = \mathfrak{X}^{-1}(p)$, only manifests itself in the gauge transformation (70). As the trace is taken over the auxiliary space $V_0 = \pi_w^p$ in (106), the matrix elements of the operator $Q_p(z)$ are therefore functions on the hypersurface (36). Identifying $p = \varphi(\xi, \zeta, \lambda)$ gives an explicit prescription how to construct the auxiliary matrix via the representations (43) and (55). In the following, the 'coordinate free' notation (106) will be used but for explicit calculations it will be understood that we invoke the coordinate map (47). The parameter μ_p corresponds to the solution of (86) satisfying (87) in order to make contact with the exact sequence (25). It can be defined solely in terms of the point $p \in$ Spec Z and is 'coordinate independent',

$$p = \left(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{c} = \mu_p + \mu_p^{-1}\right) \qquad \lim_{\mathbf{x},\mathbf{y}\to 0} \mu_p^{-N'} = q^{N'}\mathbf{z}. \tag{107}$$

Finally, note that for N even we have to set $\mathbf{x} = \mathbf{y} = 0$ in order to ensure that the intertwiner (22) exists, hence (106) reduces in this case to a one-parameter family Q_p, $p = (0, 0, \mathbf{z}, -\mathbf{z}-\mathbf{z}^{-1})$.

In order to unburden the notation for the following calculations, the explicit dependence of the auxiliary matrix on the point in the hypersurface (36) or that of the parameters (ξ, ζ, λ) will be temporarily dropped. That is, for fixed but arbitrary $p \in$ Spec Z set $Q(z) \equiv Q_p(z) = Q_{\varphi(\xi,\zeta,\lambda)}(z)$ and $\pi_w^p \equiv \pi_w$. Similarly, the matrices $Q_{p'}(z)$, $Q_{p''}(z)$ with the parameters $\varphi^{-1}(p) = (\xi', \zeta', \lambda')$, $\varphi^{-1}(p'') = (\xi'', \zeta'', \lambda'')$ given in equations (91) and (102) will simply be written as $Q'(z)$, $Q''(z)$ and $\pi_w^{p'} \equiv \pi'_w$, $\pi_w^{p''} \equiv \pi''_w$. Furthermore, we set $\mu' = \mu_{\xi',\zeta',\lambda'}$ and $\mu'' = \mu_{\xi'',\zeta'',\lambda''}$ which according to (93), (104) and (87) are given by

$$\mu' = q\mu \qquad \text{and} \qquad \mu'' = q^{-1}\mu. \tag{108}$$

We start the derivation of the functional equation (24) by considering the operator product of the auxiliary and transfer matrix which can be written as

$$Q(z)T(z) = \operatorname*{Tr}_{\pi_w \otimes \pi_z^0} L_{\pi_w,M}(z/\mu) R_{\pi_z^0,M}(z) \cdots L_{\pi_w,1}(z/\mu) R_{\pi_z^0,1}(z). \tag{109}$$

As the L-matrix is an intertwiner, it has a non-trivial kernel only when $w = \mu^{\pm 1}$, i.e., when the tensor product becomes reducible. In particular, one has

$$L(\mu^{-1})|_{\iota\pi'_{wq}} \equiv 0. \tag{110}$$

Here $\iota : \pi'_{wq} \hookrightarrow \pi_w \otimes \pi_z^0$ is the inclusion map (88). The above relation can be calculated explicitly. (The other solution $w = \mu$ is related by spin-reversal (cf equation (76) and (129)).) As a consequence of this observation and from the identity (19) for the three-fold tensor product

$$\underset{1}{\pi_w} \otimes \underset{2}{\pi_z^0} \otimes \underset{3}{\pi_1^0}$$

we can conclude that the operator products

$$L_{13}(w = z/\mu) R_{23}(z)$$

in expression (109) leave the image of the representation space π'_{wq} under the inclusion map (88) invariant. Suppose the following equations are satisfied:

$$L_{13}(w) R_{23}(z) (\iota \otimes 1) = \phi_1(z,q) (\iota \otimes 1) L'(wq) \tag{111}$$

$$(\tau \otimes 1) L_{13}(w) R_{23}(z) = \phi_2(z,q) L''(w/q) (\tau \otimes 1). \tag{112}$$

Here $\iota : \pi'_{wq} \hookrightarrow \pi_w \otimes \pi_z^0$ is again the inclusion map (88) and $\tau : \pi_w \otimes \pi_z^0 \to \pi''_{w/q}$ the projection (99). Then a functional equation of the following form must hold:

$$\begin{aligned} Q(z)T(z) &= \operatorname*{Tr}_{\pi_w \otimes \pi_z^0} L_{\pi_w,M}(z/\mu) R_{\pi_z^0,M}(z) \cdots L_{\pi_w,1}(z/\mu) R_{\pi_z^0,1}(z) \\ &= \phi_1(z,q) Q'(zq^2) + \phi_2(z,q) Q''(zq^{-2}) \qquad \mu' = q\mu \quad \mu'' = q^{-1}\mu. \end{aligned} \tag{113}$$

It remains to prove (111) and (112) and to derive the explicit form of the coefficients functions ϕ_1, ϕ_2. From the definition of the R-matrix (4), one explicitly calculates the following identities:

$$\begin{aligned} L_{13} R_{23} X_n \otimes \uparrow &= \{(a\alpha_n A v_{n+1} + c\beta_n B v_n) \otimes \uparrow + b\beta_n A v_n \otimes \downarrow\} \otimes \uparrow \\ &\quad + \{(a\alpha_n C v_{n+1} + c\beta_n D v_n) \otimes \uparrow + b\beta_n C v_n \otimes \downarrow\} \otimes \downarrow \\ &= \phi_1 (\iota \otimes 1)(A' v'_n \otimes \uparrow + C' v'_n \otimes \downarrow) \end{aligned}$$

$$\begin{aligned} L_{13} R_{23} X_n \otimes \downarrow &= \{b\alpha_n B v_{n+1} \otimes \uparrow + (c'\alpha_n A v_{n+1} + a\beta_n B v_n) \otimes \downarrow\} \otimes \uparrow \\ &\quad + \{b\alpha_n D v_{n+1} \otimes \uparrow + (c'\alpha_n C v_{n+1} + a\beta_n D v_n) \otimes \downarrow\} \otimes \downarrow \\ &= \phi_1 (\iota \otimes 1)(B' v'_n \otimes \uparrow + D' v'_n \otimes \downarrow) \end{aligned}$$

and

$$\begin{aligned} (\tau \otimes 1) L_{13} R_{23} Y_n \otimes \uparrow &= (\tau \otimes 1)(a\gamma_n A v_n \otimes \uparrow \otimes \uparrow + a\gamma_n C v_n \otimes \uparrow \otimes \downarrow) \\ &= \phi_2 (A'' v''_n \otimes \uparrow + C'' v''_n \otimes \downarrow) \\ (\tau \otimes 1) L_{13} R_{23} Y_n \otimes \downarrow &= (\tau \otimes 1)(b\gamma_n B v_n \otimes \uparrow + c'\gamma_n A v_n \otimes \downarrow) \otimes \uparrow \\ &\quad + (\tau \otimes 1)(b\gamma_n D v_n \otimes \uparrow + c'\gamma_n C v_n \otimes \downarrow) \otimes \downarrow \\ &= \phi_2 (B'' v''_n \otimes \uparrow + D'' v''_n \otimes \downarrow). \end{aligned}$$

These relations translate into equations involving the coefficients in (89) and (100). The solution consistent with all the equations is for the intertwiner (68)

$$N \text{ even}: \phi_1(z,q) = b(z,q)q^{-\frac{1}{2}}\rho_-/\rho'_- \qquad \phi_2(z,q) = a(z,q)q^{\frac{1}{2}}\rho_-/\rho''_- \tag{114}$$

and for the alternative L-matrix (71)

$$N \text{ odd}: \phi_1(z,q) = b(z,q)q^{\frac{N-1}{2}}\rho_-/\rho'_- \qquad \phi_2(z,q) = a(z,q)q^{\frac{1-N}{2}}\rho_-/\rho''_-. \tag{115}$$

Here a, b are the Boltzmann weights (5) of the six-vertex R-matrix and $\rho_-, \rho'_-, \rho''_-$ the normalization functions (69) of the intertwiners associated with the tensor products $\pi_w \otimes \pi_1^0$, $\pi'_{wq} \otimes \pi_1^0$ and $\pi''_{w/q} \otimes \pi_1^0$. If one wants to eliminate the powers of q the normalization functions should be set to

$$\rho_\pm(w,q) = q^{\frac{1\pm1}{2}} w^{\pm\frac{1}{2}} \Rightarrow \phi_1(z,q) = b(z,q), \phi_2(z,q) = a(z,q). \tag{116}$$

For N odd the correct square root has to be chosen such that $w'^{\frac{1}{2}} = q^{\frac{1-N}{2}} w^{\frac{1}{2}}$ and $w''^{\frac{1}{2}} = q^{\frac{N-1}{2}} w^{\frac{1}{2}}$. This gives the desired functional equation (24) between the transfer matrix (3) and the auxiliary matrix (106).

Again it needs to be emphasized that the three auxiliary matrices appearing in the functional equation are not equivalent in general. In fact, for N odd the values of the central elements (33) agree for all three representations π_w, π'_{wq} and $\pi''_{w/q}$ but the values of the Casimir element are different according to the identities (86), (93) and (104). Setting all three Casimir elements equal leads to the condition

$$\pi_N^{\xi,\zeta,\lambda}(\mathbf{c}) = \pi_N^{\xi',\zeta',\lambda'}(\mathbf{c}) = \pi_N^{\xi'',\zeta'',\lambda''}(\mathbf{c}) \Rightarrow \mu^2 = -q = -q^{-1}$$

implying that $q = \pm 1$ which is excluded from our construction. Thus, there is a genuine difference between the functional equation (9) and the one derived here on the basis of representation theory. Nonetheless, we may construct now in an obvious manner solutions to Baxter's functional equation (9). Recall from (41) that for a given point $p \in \operatorname{Spec} Z$, the auxiliary matrices associated with the fibre over p_* are given by

$$Q_\ell(z) \equiv Q_{p_\ell}(z) \qquad p_\ell = (\mathbf{x}, \mathbf{y}, \mathbf{z}, q^\ell\mu + \mu^{-1}q^{-\ell}) \qquad \ell \in \mathbb{Z}_N, \, p_0 \equiv p. \tag{117}$$

The functional equation (24) is now rewritten as

$$Q_\ell(z)T(z) = \phi_1(z)^M Q_{\ell+1}(zq^2) + \phi_2(z)^M Q_{\ell-1}(zq^{-2}). \tag{118}$$

If we now sum over all the points in the fibre one immediately deduces that the operator

$$Q_{p_*}^{(s)}(z) \equiv \sum_{\ell\in\mathbb{Z}_N} q^{-s\ell} Q_\ell(z) \qquad p_* = (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in \operatorname{Spec} Z_0, \, s \in \mathbb{Z}_N \tag{119}$$

provides a valid solution to Baxter's functional equation (9) with a slight modification of the coefficient functions ϕ_1, ϕ_2,

$$Q_{p_*}^{(s)}(z)T(z) = \phi_1(z)^M q^s Q_{p_*}^{(s)}(zq^2) + \phi_2(z)^M q^{-s} Q_{p_*}^{(s)}(zq^{-2}). \tag{120}$$

As indicated this solution lives on $\operatorname{Spec} Z_0$ rather than $\operatorname{Spec} Z$ (cf (37)).

For N even one has to set $\mathbf{x} = \mathbf{y} = 0$ and the functional equation (24) in terms of the coordinate map (47) now reads

$$N \text{ even}: Q_\lambda(z)T(z) = \phi_1(z)^M Q_{\lambda q^{-1}}(zq^2) + \phi_2(z)^M Q_{\lambda q}(zq^{-2}) \qquad Q_\lambda \equiv Q_{\varphi(0,0,\lambda)}. \tag{121}$$

The auxiliary matrices are again not equivalent as they belong to different root of unity representations. This time not only the values of the Casimir elements are different but also the values of the central element $\mathbf{z}$ (cf equations (92) and (103)). We therefore have to sum over two fibres, i.e., $\mathbb{Z}_N$ instead of $\mathbb{Z}_{N'}$, to obtain a solution to (9). Thus, the expression (119) applies to all roots of unity when setting $Q_\ell(z) \equiv Q_{\varphi(0,0,\lambda q^\ell)}(z)$.

The following cautious remark with regard to the solutions (119) of Baxter's functional equation (9) must be made. Depending on the length of the spin-chain as well as the spin-sector, it can happen that these solutions are trivial, i.e., they might sum up to zero. We will investigate this below for two examples. There we verify that the solutions (119) are non-trivial in the spin-zero sector of the four-chain, where we compare them with Baxter's expression (18). This shows that at roots of unity one can find (at least in certain sectors) solutions to (9) which are of a simpler form than (17), namely finite sums of the expression (16). However, from the construction it is clear that the auxiliary matrices (106) defined on Spec Z should be regarded as the fundamental objects in the present setting.

5.1. Transformation properties of the auxiliary matrix

In this section, the transformation properties of the auxiliary matrix related to the symmetries of the six-vertex transfer matrix (3) are investigated. The first transformation law involves the total spin (6) and is a direct consequence from the intertwining relation (22). As the coproduct and opposite coproduct coincide for the Cartan elements, one has

$$L^p(w)\pi^p(k)\otimes q^{\sigma^z} = \pi^p(k)\otimes q^{\sigma^z} L^p(w)$$

which in turn implies

$$\left[Q_p(z), q^{\sigma^z}\otimes q^{\sigma^z}\cdots\otimes q^{\sigma^z}\right] = 0. \tag{122}$$

Hence, the auxiliary matrix flips at most multiples of N' spins. This property is due to non-vanishing contributions in the trace in (106) containing the matrices $B^{N'}, C^{N'}$ which depend on the central elements $\mathbf{x}, \mathbf{y}$. The only other non-vanishing contributions contain the operators B, C in pairs and thus do not change the total spin. Consequently, we have the transformation law

$$\mathrm{e}^{tS^z} Q_p(z)\,\mathrm{e}^{-tS^z} = Q_{\mathrm{e}^{tS^z}p}(z) \tag{123}$$

where the following map on the hypersurface (36) has been introduced:

$$\mathrm{Spec}\,Z \ni p = (\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{c}) \to \mathrm{e}^{tS^z} p := (\mathrm{e}^{-tN}\mathbf{x},\ \mathrm{e}^{tN}\mathbf{y},\mathbf{z},\mathbf{c}). \tag{124}$$

For nilpotent representations, $\mathbf{x} = \mathbf{y} = 0$, the auxiliary matrices obviously commute with the total spin operator.

Next let us investigate the transformation of the auxiliary matrices under the action of $\mathfrak{S}$ defined in (7). From the following simple transformation of the L-matrix:

$$(1\otimes\sigma^z)L(1\otimes\sigma^z) = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix} \tag{125}$$

and the fact that the only non-vanishing terms in the trace (106) contain the step operators B, C either in pairs or to the power N when N is odd, one deduces the transformation property

$$\mathfrak{S} Q_p(z)\mathfrak{S} = Q_{\mathfrak{S}p}(z) \tag{126}$$

with

$$\mathrm{Spec}\,Z \ni p = (\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{c}) \to \mathfrak{S}p := (-\mathbf{x},-\mathbf{y},\mathbf{z},\mathbf{c}). \tag{127}$$

This includes the case N even where $\mathbf{x} = \mathbf{y} = 0$. The transformation behaviour (126) together with the second identity in (8) allows us to discuss the case of even primitive roots of unity $q^{2N'} = 1$ with N' odd and cyclic representations. Performing the replacement $q \to -q$, we obviously recover the case of odd roots of unity and can therefore conclude for even M,

$$Q_p(z,-q)\mathfrak{S}T(z,q) = b(z,q)^M Q_{p'}(zq^2,-q) + a(z,q)^M Q_{p''}(zq^{-2},-q). \tag{128}$$

Here the trivial relations $b(z,-q) = -b(z,q), a(z,-q) = a(z,q)$ have been used. The operators in this functional equation do not in general commute with each other due to (126). Nevertheless, this functional equation might be useful to gain insight in the different structure encountered for even and odd roots of unity.

Finally, we investigate the behaviour under spin-reversal employing the previous investigations of section 3.1. There we already saw that the spin-reversal transformation induces the mapping (cf (82))

$$\operatorname{Spec} Z \ni p = (\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{c}) \to \Re p := (\mathbf{y},\mathbf{x},\mathbf{z}^{-1},\mathbf{c}).$$

According to the definition (106), we have $w = z/\mu_p$ where μ_p is given by the quadratic equation (107). While the value of the Casimir element stays invariant under spin-reversal, we now need to switch to the other solution μ_p^{-1} satisfying

$$\lim_{\mathbf{x},\mathbf{y}\to 0} \mu_{\Re p}^{-N'} = \lim_{\mathbf{x},\mathbf{y}\to 0} \left(\mu_p^{-1}\right)^{-N'} = \mathbf{z}^{-1}. \tag{129}$$

This follows directly from (82). Hence, we obtain the following transformation law for the auxiliary matrix:

$$\begin{aligned} \Re Q_p(z)\Re &= \operatorname*{Tr}_{V_0=\pi_w^p} \sigma_M^x L_{0M}^p(w)\sigma_M^x \cdots \sigma_1^x L_{01}^p(w)\sigma_1^x \\ &= \mathbf{z}^M \operatorname*{Tr}_{V_0=\pi_w^{\Re p}} w^{-\sigma_M^z/2} L_{0M}^{\Re p}(w) w^{\sigma_M^z/2} \cdots w^{-\sigma_1^z/2} L_{01}^{\Re p}(w) w^{\sigma_1^z/2} \\ &= \mathbf{z}^M w^{-S^z} Q_{\Re p}\left(z\mu_p^{-2}\right) w^{S^z} \qquad w = z/\mu_p. \end{aligned} \tag{130}$$

This transformation law simplifies for nilpotent representations, $\mathbf{x},\mathbf{y} = 0$, using the coordinate map (47) to

$$\Re Q_\lambda(z)\Re = \lambda^{N'M} Q_{\lambda^{-1}q^{-2}}(z\lambda^2 q^2). \tag{131}$$

For even roots of unity and the solution (68), the factor $\lambda^{N'M} = \mathbf{z}^M$ has to be omitted. Note that spin-reversal symmetry is only broken for spin-chains which are sufficiently long.

The last transformation law we are going to derive involves auxiliary matrices with inverted arguments. Setting for simplicity $\rho_+ = wq, \rho_- = 1$ in (68) respectively (71) it is straightforward to verify the identity

$$(1\otimes\sigma^x)L^p(w)(1\otimes\sigma^x) = -wq\left(1\otimes(-wq)^{-\frac{\sigma^z}{2}}\right)L^p(w^{-1}q^{-2})^{t_2}\left(1\otimes(-wq)^{+\frac{\sigma^z}{2}}\right). \tag{132}$$

Here the superscript t_2 denotes transposition with respect to the second factor. Consequently, the auxiliary matrix obeys the relation

$$\Re Q_p(z)\Re = (-wq)^M(-w)^{-S^z} Q_p\left(z^{-1}q^{-2}\mu_p^2\right)^t(-w)^{+S^z} \qquad w = z/\mu_p. \tag{133}$$

This quite complicated looking transformation behaviour again simplifies for nilpotent representations where spin-conservation is restored.

5.1.1. Principal gradation. For completeness, let us now consider the principal gradation (81). According to the discussion in section 3.2 (cf (84)), the corresponding auxiliary matrix is calculated to be

$$\begin{aligned} \mathcal{Q}_p(x) &= \operatorname*{Tr}_0 \mathcal{L}_{0M}\left(x/\mu^{\frac12}\right)\cdots\mathcal{L}_{01}\left(x/\mu^{\frac12}\right) \\ &= \left(x/\mu^{\frac12}\right)^{-S^z} \operatorname*{Tr}_0 L_{0M}(x^2/\mu)\cdots L_{01}(x^2/\mu)\left(x/\mu^{\frac12}\right)^{S^z} \\ &= \left(x/\mu^{\frac12}\right)^{-S^z} Q_p(x^2)\left(x/\mu^{\frac12}\right)^{S^z}. \end{aligned} \tag{134}$$

Exploiting the transformation law (123), this allows us to rewrite all the previous relations for the auxiliary matrix (106) in terms of the principal gradation. For example, exploiting (6) the functional equation with the transfer matrix now reads

$$\mathcal{Q}_p(x)T(x^2) = \phi_1(x^2,q)^M \mathcal{Q}_{p'}(xq) + \phi_2(x^2,q)^M \mathcal{Q}_{p''}(xq^{-1}) \qquad z = x^2. \tag{135}$$

For nilpotent representations, both gradations are obviously equivalent as the auxiliary matrix then conserves the total spin.

5.2. Commutation relations

In order to make contact between the functional equation (24) and to derive the Bethe ansatz equations (2) for the six-vertex model, one needs to ensure that all the operators in (24) commute with each other. The commutation of the transfer matrix with the different auxiliary matrices is immediate from the existence of the intertwiner (22) and (19). We have already seen that there are no restrictions on $p \in \text{Spec } Z\backslash D$ when *N* is odd. For *N* even we have to set $\mathbf{x} = \mathbf{y} = 0$ as mentioned before.

Employing the same argument to guarantee the commutation of the auxiliary matrices in (24), let us verify whether the necessary existence criteria (58) are satisfied for the corresponding intertwiners. It is worthwhile doing this for two arbitrary points $p = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{c})$ and $\bar{p} = (\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{z}}, \bar{\mathbf{c}})$. We are looking for an operator such that

$$S_{p\bar{p}}(w,\bar{w})\big(\pi_w^p \otimes \pi_{\bar{w}}^{\bar{p}}\big)\Delta(x) = \big[\big(\pi_w^p \otimes \pi_{\bar{w}}^{\bar{p}}\big)\Delta^{\text{op}}(x)\big]S_{p\bar{p}}(w,\bar{w}), x \in U_q(\widetilde{sl}_2). \tag{136}$$

The case of even roots of unity is trivial as only nilpotent representations occur and one finds that the necessary requirements (58) are met for arbitrary values of the spectral parameters $w, \bar{w}$. For *N* odd cyclic representations are allowed and one finds

$$\begin{aligned} &\mathbf{x} + \mathbf{z}\bar{\mathbf{x}} = \bar{\mathbf{x}} + \mathbf{x}\bar{\mathbf{z}} && \mathbf{y}\bar{\mathbf{z}}^{-1} + \bar{\mathbf{y}} = \mathbf{z}^{-1}\bar{\mathbf{y}} + \mathbf{y} \\ &\mathbf{x}\bar{\mathbf{z}} + \bar{\mathbf{x}}(w/\bar{w})^N = \mathbf{z}\bar{\mathbf{x}}(w/\bar{w})^N + \mathbf{x} && (w/\bar{w})^N\mathbf{y} + \bar{\mathbf{y}}\mathbf{z}^{-1} = \bar{\mathbf{y}} + \mathbf{y}\bar{\mathbf{z}}^{-1}(w/\bar{w})^N. \end{aligned} \tag{137}$$

From these equations, one deduces that $(w/\bar{w})^N = 1$ needs to hold unless $\mathbf{z} = \bar{\mathbf{z}} = 1$. According to (92) and (103), the values of the central elements (33) coincide for all three representations in the exact sequence (25). Hence, the criteria are met for all three auxiliary matrices in (24) respectively (118). Strictly speaking these requirements are necessary for the existence of an intertwiner but not sufficient. However, employing the results of the important paper [24], the intertwiner (136) can be shown to exist for $q^N = 1$ with *N* odd.

5.2.1. Connection with the chiral Potts model.

Bazhanov and Stroganov pointed out [24] that the Boltzmann weights of the chiral Potts model [37, 38] solve the Yang–Baxter algebra of the *L*-operators at roots of unity. As the solution in [24] is given in a particular root-of-unity representation different from (43), it is helpful to briefly review the results and make the connection with evaluation representations of $U_q(\widetilde{sl}_2)$ explicit.

Denote by $\{v_n\}_{n\in\mathbb{Z}_N}$ as before the standard basis in $\mathbb{C}^N$ and define the operators

$$Zv_n = q^{-n}v_n \qquad Xv_n = v_{n+1} \qquad n \in \mathbb{Z}_N. \tag{138}$$

Then the solution to the Yang–Baxter algebra (19) found in [24] is given by (setting $A = Z, B = Z^{-1}, C = X$ with respect to the notation used in equations (2.12) and (2.13) in [24])

$$\tilde{L}(w) = \begin{pmatrix} w^{\frac{1}{2}}d_+Z + w^{-\frac{1}{2}}d_-Z^{-1} & w^{\frac{1}{2}}(g_+Z^{-1} + g_-Z)X \\ w^{-\frac{1}{2}}(h_+Z^{-1} + h_-Z)X^{-1} & w^{\frac{1}{2}}f_+Z^{-1} + w^{-\frac{1}{2}}f_-Z \end{pmatrix} \tag{139}$$

where the six coefficients $\chi = \{d_+, d_-, f_+, f_-, g_+, g_-\}$ are independent complex parameters and $h_\pm = f_\pm d_\mp / g_\pm$. Under a suitable choice of the parameters, this solution can be identified as an intertwiner of the Hopf algebra $\breve{U}_q(\widetilde{sl}_2)$. Given a representation π of the finite subalgebra $\breve{U}_q(sl_2)$, the solution to the intertwiner equation (22) for the associated evaluation representation now is

$$\breve{L}(w) = \begin{pmatrix} \breve{\rho}_+\pi(t) - \breve{\rho}_-\pi(t)^{-1} & \breve{\rho}_+(q-q^{-1})q^{-\frac{1}{2}}\pi(\breve{f}) \\ \breve{\rho}_-(q-q^{-1})q^{\frac{1}{2}}\pi(\breve{e}) & \breve{\rho}_+\pi(t)^{-1} - \breve{\rho}_-\pi(t) \end{pmatrix} \qquad \breve{\rho}_+/\breve{\rho}_- = wq. \tag{140}$$

The additional factors of $q^{\pm\frac{1}{2}}$ in the off-diagonal matrix elements in comparison to (68) are explained by choosing (21) as representation of $\breve{U}_q(\widetilde{sl}_2)$ instead of $U_q(\widetilde{sl}_2)$. Invoking the following cyclic representation:

$$\pi(\breve{e}) = s_0^{-1}\frac{s_1 Z^{-1} - s_1^{-1} Z}{q - q^{-1}} X^{-1} \qquad \pi(\breve{f}) = s_0 \frac{s_2 Z - s_2^{-1} Z^{-1}}{q - q^{-1}} X \qquad \pi(t) = \left(\frac{s_1}{s_2}\right)^{-\frac{1}{2}} Z \tag{141}$$

with $s_0, s_1, s_2 \in \mathbb{C}^\times$ and central values

$$\begin{aligned} &\pi(\mathbf{x}) = \left(\frac{s_1}{s_2}\right)^{-\frac{N}{2}} \left(s_1^N - s_1^{-N}\right) s_0^{-N} \qquad \pi(\mathbf{y}) = \left(\frac{s_1}{s_2}\right)^{\frac{N}{2}} \left(s_2^N - s_2^{-N}\right) s_0^N \\ &\pi(\mathbf{z}) = (s_1/s_2)^{-N} \qquad \pi(\mathbf{c}) = q^{-1}s_1 s_2 + q(s_1 s_2)^{-1} \end{aligned} \tag{142}$$

the solution (139) can be identified as the intertwiner (140) upon setting

$$\breve{\rho}_\pm = q^{\pm\frac{1}{2}} w^{\pm\frac{1}{2}} \qquad d_\pm = -q^{\pm 1} f_\mp = \pm q^{\pm\frac{1}{2}} (s_1/s_2)^{\mp\frac{1}{2}} \qquad g_\pm = \mp s_0 s_2^{\mp 1}, h_\pm = \pm s_0^{-1} s_1^{\pm 1}. \tag{143}$$

The three additional parameters of (139) in comparison to the solution (68) can be accounted for as follows. First note that the ratio of the coefficient functions (69) is arbitrary if one requires the L -matrix only to satisfy (19). The remaining two parameters can be understood in terms of Drinfel'd's quantum double construction [30]. They fix the value of two additional central elements which arise when one considers the quantum double of the upper Borel subalgebra (see, e.g., [35] for an explanation in the context of the chiral Potts model). The solution (68) is obtained when setting these central elements to one.

In [24], the authors showed by construction that the operator products

$$\tilde{L}_{13}(w_1, \chi_1)\tilde{L}_{23}(w_2, \chi_2) \qquad \text{and} \qquad \tilde{L}_{23}(w_2, \chi_2)\tilde{L}_{13}(w_1, \chi_1)$$

are equivalent provided the parameter sets χ_1, χ_2 share three common invariants (cf equations (4.4) and (4.5) on page 809 in [24]). Employing the identification (143), these invariants can be expressed in terms of the central elements (33) in the representation (141),

$$\begin{aligned} \Gamma_1 &= \frac{\left(d_+^N - f_+^N\right)\left(d_-^N - f_-^N\right)}{\left(g_+^N + g_-^N\right)\left(h_+^N + h_-^N\right)} = \frac{\left(1 - s_2^N/s_1^N\right)\left(1 - s_1^N/s_2^N\right)}{\left(s_1^N - s_1^{-N}\right)\left(s_2^N - s_2^{-N}\right)} = \frac{(1-\pi(\mathbf{z})^{-1})(1-\pi(\mathbf{z}))}{\pi(\mathbf{x})\pi(\mathbf{y})} \\ \Gamma_2 &= w^{-N}\frac{d_-^N - f_-^N}{d_+^N - f_+^N} = w^{-N}\frac{\left(s_1^N/s_2^N - 1\right)}{\left(s_1^N/s_2^N - 1\right)} = w^{-N} \\ \Gamma_3 &= w^{-N}\frac{h_+^N + h_-^N}{g_+^N + g_-^N} = -w^{-N}\frac{s_0^{-N}\left(s_1^N - s_1^{-N}\right)}{s_0^N\left(s_2^N - s_2^{-N}\right)} = -w^{-N}\pi(\mathbf{z}^{-1})\pi(\mathbf{x})/\pi(\mathbf{y}). \end{aligned}$$

Comparing with (137), one now deduces that the intertwiner (136) indeed exists and is a special case of the chiral Potts model. As demonstrated the auxiliary matrix (106) is independent of the choice of the root-of-unity representation, hence we can conclude that the following auxiliary matrices commute

$$[Q_p(z), Q_{\bar{p}}(\bar{z})] = 0 \qquad \frac{\mathbf{x}}{1-\mathbf{z}} = \frac{\bar{\mathbf{x}}}{1-\bar{\mathbf{z}}}, \frac{\mathbf{y}}{1-\mathbf{z}^{-1}} = \frac{\bar{\mathbf{y}}}{1-\bar{\mathbf{z}}^{-1}} \qquad z^N = (z\mu/\bar{\mu})^N. \quad (144)$$

Clearly for any two points $p = p_\ell$, $\bar{p} = p_k$, $k, \ell \in \mathbb{Z}$ in the fibre (41), the above conditions are satisfied and all the auxiliary matrices (117) evaluated at the same spectral value z commute with each other. In particular, this allows us to write the functional equation (24) in terms of eigenvalues and to derive the Bethe ansatz equations (2).

5.2.2. Comment on previous solutions to Baxter's functional equation. Note that auxiliary matrices for the functional equation (9) and not (24) have been considered in [24] and [26]. In [24] (p 805), (see also section 3 in [26]), it is stated that one is then forced to make the specific choice

$$\{d_+, d_-, f_+, f_-, g_+, g_-, h_+, h_-\} \to \left\{a, b, cw^{-\frac{1}{2}}, dw^{\frac{1}{2}}, \lambda b, \lambda a, \lambda c w^{-\frac{1}{2}}, \lambda d w^{\frac{1}{2}}\right\}$$

of the parameters in (139). The corresponding five-parameter auxiliary matrices are built from the operators

$$\tilde{L}(w) = \begin{pmatrix} w^{\frac{1}{2}} aZ + w^{-\frac{1}{2}} bZ^{-1} & w^{\frac{1}{2}}\lambda(bZ^{-1} + aZ)X \\ w^{-1}\lambda(cZ^{-1} + wdZ)X^{-1} & cZ^{-1} + dZ \end{pmatrix}. \qquad (145)$$

Note, that due to the additional spectral variable dependence in the parameters this L-matrix does not satisfy the Yang–Baxter equation (19). (Because $Q_{R,L}$ in Baxter's construction need not commute with the transfer matrix, this poses no problem.) Moreover, the above solution (145) cannot be interpreted in terms of evaluation representations of the quantum loop algebra. The auxiliary matrices considered in [24, 26] are assumed to be of the form (16) and to solve (9). This corresponds to Baxter's construction procedure for the eight- [15–19] and six-vertex model [19] which is different from the one applied here. We already saw that the solution to (9) in the present framework is given as a sum of the expressions (16) (cf (119)).

6. Two simple examples: $N = 3$, $M = 3, 4$

In this section, the specific examples $N = 3$, $M = 3, 4$ are considered in order to illustrate the construction procedure of the auxiliary matrix (106) and to demonstrate the working of the functional equation (24). The matrices in (24) are diagonalized and it is shown explicitly that the additional parameter dependence of the auxiliary matrices then drops out of the functional equation. This must be the case as the eigenvalues of the transfer matrix (3) and the Bethe ansatz equations (2) only depend on the variables z and q. In particular, it is shown that the centre of the complete N-strings (14) describing the degenerate eigenstates of the six-vertex model are given in terms of the central elements of the quantum group. Furthermore, we will compare the auxiliary matrix (106) for the four chain with Baxter's expression (18) in the spin-zero sector.

Invoking the representation (43), the Chevalley generators of the quantum group are

$$\pi_3^{\xi,\zeta,\lambda}(k) = \lambda \begin{pmatrix} 1 & & \\ & q^{-2} & \\ & & q^{-4} \end{pmatrix} \qquad \pi_3^{\xi,\zeta,\lambda}(f) = \begin{pmatrix} 0 & 0 & \zeta \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\pi_3^{\xi,\zeta,\lambda}(e) = \begin{pmatrix} 0 & \xi\zeta + \frac{\lambda-\lambda^{-1}}{q-q^{-1}} & 0 \\ 0 & 0 & \xi\zeta - q^{N'}\frac{\lambda q^{-1}-\lambda^{-1}q}{q-q^{-1}} \\ \xi & 0 & 0 \end{pmatrix}. \qquad (146)$$

Using these expression, one can now explicitly write down the intertwiner (71) and the auxiliary matrix (106). Nonetheless, all expressions will be given in terms of the values of the central elements, independent of the representation used. This emphasizes the aforementioned fact that the matrix elements of (106) are functions on the hypersurface (36).

6.1. The $M = 3$ spin-chain

We start with $M = 3$ as it is the minimum length of the spin-chain required for the breaking of spin-reversal symmetry and spin-conservation when $N = 3$. One finds the following non-vanishing matrix elements of (106):

$$\begin{aligned}
&2S^z = \pm 3 : Q^{\uparrow\uparrow\uparrow}_{\uparrow\uparrow\uparrow} = \mathrm{Tr}\, A^3 \qquad Q^{\uparrow\uparrow\uparrow}_{\downarrow\downarrow\downarrow} = \mathrm{Tr}\, B^3 \qquad Q^{\downarrow\downarrow\downarrow}_{\uparrow\uparrow\uparrow} = \mathrm{Tr}\, C^3 \qquad Q^{\downarrow\downarrow\downarrow}_{\downarrow\downarrow\downarrow} = \mathrm{Tr}\, D^3 \\
&2S^z = \pm 1 : Q^{\downarrow\uparrow\uparrow}_{\downarrow\uparrow\uparrow} = Q^{\uparrow\downarrow\uparrow}_{\uparrow\downarrow\uparrow} = Q^{\uparrow\uparrow\downarrow}_{\uparrow\uparrow\downarrow} = \mathrm{Tr}\, A^2 D \qquad Q^{\uparrow\downarrow\downarrow}_{\uparrow\downarrow\downarrow} = Q^{\downarrow\uparrow\downarrow}_{\downarrow\uparrow\downarrow} = Q^{\downarrow\downarrow\uparrow}_{\downarrow\downarrow\uparrow} = \mathrm{Tr}\, D^2 A \\
&2S^z = 1 : Q^{\uparrow\downarrow\uparrow}_{\downarrow\uparrow\uparrow} = Q^{\uparrow\uparrow\downarrow}_{\uparrow\downarrow\uparrow} = Q^{\downarrow\uparrow\uparrow}_{\uparrow\uparrow\downarrow} = \mathrm{Tr}\, ACB \qquad Q^{\uparrow\uparrow\downarrow}_{\downarrow\uparrow\uparrow} = Q^{\downarrow\uparrow\uparrow}_{\uparrow\downarrow\uparrow} = Q^{\uparrow\downarrow\uparrow}_{\uparrow\uparrow\downarrow} = \mathrm{Tr}\, ABC \\
&2S^z = -1 : Q^{\downarrow\downarrow\uparrow}_{\uparrow\downarrow\downarrow} = Q^{\downarrow\downarrow\uparrow}_{\downarrow\uparrow\downarrow} = Q^{\uparrow\downarrow\downarrow}_{\downarrow\downarrow\uparrow} = \mathrm{Tr}\, DBC \qquad Q^{\downarrow\downarrow\uparrow}_{\uparrow\downarrow\downarrow} = Q^{\uparrow\downarrow\downarrow}_{\downarrow\uparrow\downarrow} = Q^{\downarrow\uparrow\downarrow}_{\downarrow\downarrow\uparrow} = \mathrm{Tr}\, DCB.
\end{aligned} \tag{147}$$

To simplify the notation we have dropped the dependence on the point in the hypersurface and the spectral variable. The matrix elements are defined according to the convention $Q(z)|\underline{\alpha}\rangle = \sum_{\underline{\beta}} Q(z)^{\underline{\beta}}_{\underline{\alpha}} |\underline{\beta}\rangle$. From the matrix elements, we infer that each of the sectors $S^z = \pm 1/2$ is mapped into itself. Thus we need only diagonalize the matrices

$$Q|_{S^z=\pm 3/2} = \begin{pmatrix} \mathrm{Tr}\, A^3 & \mathrm{Tr}\, B^3 \\ \mathrm{Tr}\, C^3 & \mathrm{Tr}\, D^3 \end{pmatrix} \qquad \text{and} \qquad Q|_{S^z=1/2} = \begin{pmatrix} \mathrm{Tr}\, A^2 D & \mathrm{Tr}\, ABC & \mathrm{Tr}\, ACB \\ \mathrm{Tr}\, ACB & \mathrm{Tr}\, A^2 D & \mathrm{Tr}\, ABC \\ \mathrm{Tr}\, ABC & \mathrm{Tr}\, ACB & \mathrm{Tr}\, A^2 D \end{pmatrix}. \tag{148}$$

The matrix $Q|_{S^z=-1/2}$ is obtained by exploiting the transformation law (130). The matrix elements turn out to take an algebraically simpler form when the following choice of the normalization functions (69) is made:

$$\rho_\pm = 3^{-\frac{1}{3}} (wq)^{\frac{1\pm 1}{2}}.$$

As $M = N = 3$, this does not change the form of the functional equation (24) (cf (113) and (115)). The matrix elements are calculated to

$$\begin{aligned}
&\mathrm{Tr}\, A^3 = w^3 \mathbf{z}^2 - \mathbf{z} \qquad \mathrm{Tr}\, B^3 = w^3 \mathbf{y}\mathbf{z}^2 \qquad \mathrm{Tr}\, C^3 = \mathbf{x}\mathbf{z} \qquad \mathrm{Tr}\, D^3 = w^3 \mathbf{z} - \mathbf{z}^2 \\
&\mathrm{Tr}\, A^2 D = wq\mathbf{z}(1 - wq\mathbf{z}) \qquad \mathrm{Tr}\, ABC = w\mathbf{z}(1 - w\mathbf{z}) \qquad \mathrm{Tr}\, ACB = wq\mathbf{z}(q - w\mathbf{z}).
\end{aligned} \tag{149}$$

Here the spectral variable is set to $w = z/\mu$ with μ given by (86) and the central elements take the values detailed in (45). The eigenvalues of the auxiliary matrix in the sector $S^z = \pm 3/2$ are found to be

$$\begin{aligned}
Q_\pm(w) &= \frac{1}{2}\left(\mathrm{Tr}\, A^3 + \mathrm{Tr}\, D^3 \pm \sqrt{(\mathrm{Tr}\, A^3 - \mathrm{Tr}\, D^3)^2 + 4\, \mathrm{Tr}\, B^3\, \mathrm{Tr}\, C^3}\right) \\
&= \frac{\mathbf{z}}{2}\left((w^3 - 1)(\mathbf{z} + 1) \pm \sqrt{(\mathbf{z} - 1)^2 (w^3 + 1)^2 + 4w^3 \mathbf{x}\mathbf{y}\mathbf{z}}\right).
\end{aligned} \tag{150}$$

The zeros of these two eigenvalues form two-complete 3-string,

$$Q_\pm(w_n) = 0 \qquad w_n = q^n \left\{\mathbf{x}\mathbf{y} + \mathbf{z} + \mathbf{z}^{-1} \pm \sqrt{(\mathbf{x}\mathbf{y} + \mathbf{z} + \mathbf{z}^{-1})^2 - 4}\right\}^{\frac{1}{3}} \qquad n = 0, 1, 2. \tag{151}$$

Note that the string centre is determined by the central elements (33) and one has

$$\mathbf{xy} + \mathbf{z} + \mathbf{z}^{-1} = F_3(\mathbf{c}) = \mu^3 + \mu^{-3}.$$

Thus, the above complete 3-string simplify to $w_n = q^n \mu^{\pm 1}$. If the nilpotent limit is taken, the simple form (14) for the eigenvalues is recovered. The complete 3-string contributions cancel on both sides of the functional equation (24) due to the q-periodicity of $Q_\pm$. Thus, one obtains the corresponding eigenvalues $T_\pm(z) = a(z)^3 + b(z)^3$ of the transfer matrix (3) as required.

For the sector $S^z = 1/2$, one finds the eigenvalues

$$Q_0(w) = \mathrm{Tr}\, A^2 D + \mathrm{Tr}\, ABC + \mathrm{Tr}\, ACB \equiv 0 \tag{152}$$

$$Q_1(w) = \mathrm{Tr}\, A^2 D + q\, \mathrm{Tr}\, ABC + q^2\, \mathrm{Tr}\, ACB = 3qw\mathbf{z} \tag{153}$$

$$Q_2(w) = \mathrm{Tr}\, A^2 D + q^2\, \mathrm{Tr}\, ABC + q\, \mathrm{Tr}\, ACB = -3q^2 w^2 \mathbf{z}^2. \tag{154}$$

From the first expression, we infer that the auxiliary matrix turns out to be singular and not all eigenvalues of the transfer matrix can be calculated. The solution (119) for $s = 0$ vanishes completely. If we set $s = 1, 2$ either $Q_1^{(s)}$ or $Q_2^{(s)}$ becomes zero showing that the solution (119) has a null space of higher rank than (106).

For the remaining two eigenvalues in the sector $S^z = 1/2$ one reads off a simple and a double zero at $w = z = 0$. These correspond to 'Bethe roots at infinity' when the parametrization $z = \mathrm{e}^u q^{-1}$ is used in the Bethe ansatz equations (2). That such 'Bethe roots' can occur is a known phenomenon (see, e.g., the discussion in [10] and references therein). Note that also here the dependence on the central elements drops out of the equation (24) showing as expected that the corresponding eigenvalues of the transfer matrix

$$T_1(z) = b(z)^3 q + a(z)^3 q^2 \qquad T_2(z) = b(z)^3 q^2 + a(z)^3 q$$

are independent of the point $p = \varphi(\xi, \zeta, \lambda)$ in the hypersurface (36).

6.2. *The $M = 4$ spin-chain*

We now consider the spin-sectors $S^z = 0, -1$ in the four-chain. As all of the following matrix elements vanish:

$$Q^{\uparrow\downarrow\downarrow\downarrow}_{\uparrow\uparrow\uparrow\uparrow} = Q^{\downarrow\uparrow\downarrow\downarrow}_{\uparrow\uparrow\uparrow\uparrow} = Q^{\downarrow\downarrow\uparrow\downarrow}_{\uparrow\uparrow\uparrow\uparrow} = Q^{\downarrow\downarrow\downarrow\uparrow}_{\uparrow\uparrow\uparrow\uparrow} = \mathrm{Tr}\, AC^3 = 0 \tag{155}$$

$$Q^{\uparrow\uparrow\uparrow\uparrow}_{\uparrow\downarrow\downarrow\downarrow} = Q^{\uparrow\uparrow\uparrow\uparrow}_{\downarrow\uparrow\downarrow\downarrow} = Q^{\uparrow\uparrow\uparrow\uparrow}_{\downarrow\downarrow\uparrow\downarrow} = Q^{\uparrow\uparrow\uparrow\uparrow}_{\downarrow\downarrow\downarrow\uparrow} = \mathrm{Tr}\, AB^3 = 0 \tag{156}$$

the remaining sectors are either trivial ($S^z = \pm 2$) or related by spin-reversal ($S^z = 1$). Let us start with the spin-sector of smaller dimension, i.e., $S^z = -1$.

6.2.1. $S^z = -1$.

The auxiliary matrix in this sector is computed to be

$$Q|_{S^z=-1} = \begin{pmatrix} \mathrm{Tr}\, AD^3 & \mathrm{Tr}\, CBD^2 & \mathrm{Tr}\, BDCD & \mathrm{Tr}\, BCD^2 \\ \mathrm{Tr}\, BCD^2 & \mathrm{Tr}\, AD^3 & \mathrm{Tr}\, CBD^2 & \mathrm{Tr}\, BDCD \\ \mathrm{Tr}\, BDCD & \mathrm{Tr}\, BCD^2 & \mathrm{Tr}\, AD^3 & \mathrm{Tr}\, CBD^2 \\ \mathrm{Tr}\, CBD^2 & \mathrm{Tr}\, BDCD & \mathrm{Tr}\, BCD^2 & \mathrm{Tr}\, AD^3 \end{pmatrix} \tag{157}$$

where the basis vectors in the spin-sector have been chosen such that the first column vector is given by

$$Q^{\uparrow\downarrow\downarrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} = \mathrm{Tr}\, AD^3 \qquad Q^{\downarrow\uparrow\downarrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} = \mathrm{Tr}\, BCD^2 \qquad Q^{\downarrow\downarrow\uparrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} = \mathrm{Tr}\, BDCD \qquad Q^{\downarrow\downarrow\downarrow\uparrow}_{\uparrow\downarrow\downarrow\downarrow} = \mathrm{Tr}\, CBD^2. \tag{158}$$

Choosing as before the conventions

$$\rho_\pm = (wq)^{\frac{1\pm1}{2}} \qquad \phi_1 = bq^{\frac{N-1}{2}} \qquad \phi_2 = aq^{\frac{1-N}{2}} \qquad \mathbf{c} = \mu + \mu^{-1} \qquad w = z/\mu$$

one finds after some algebra the expressions

$$\begin{aligned} Q^{\uparrow\downarrow\downarrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} &= \operatorname{Tr} AD^3 = -9w\mathbf{z}^2(w^2+q) \\ Q^{\downarrow\uparrow\downarrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} &= \operatorname{Tr} BCD^2 = -3w\mathbf{z}^2(1+qw^2+2wq^2\mathbf{c}) \end{aligned} \tag{159}$$

$$\begin{aligned} Q^{\downarrow\downarrow\uparrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} &= \operatorname{Tr} BDCD = -3w\mathbf{z}^2(w^2+q-wq^2\mathbf{c}) \\ Q^{\downarrow\downarrow\downarrow\uparrow}_{\uparrow\downarrow\downarrow\downarrow} &= \operatorname{Tr} CBD^2 = -3w\mathbf{z}^2(q^2+q^2w^2+2wq^2\mathbf{c}). \end{aligned} \tag{160}$$

Diagonalizing the above matrix then yields the eigenvalues

$$\begin{aligned} Q_1 &= Q^{\uparrow\downarrow\downarrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} + Q^{\downarrow\uparrow\downarrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} + Q^{\downarrow\downarrow\uparrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} + Q^{\downarrow\downarrow\downarrow\uparrow}_{\uparrow\downarrow\downarrow\downarrow} \\ &= -9w\mathbf{z}^2(w^2+wq^2\mathbf{c}+q) = -9\mathbf{z}^2w(w+q^2\mu^{-1})(w+q^2\mu) \\ Q_2 &= Q^{\uparrow\downarrow\downarrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} + Q^{\downarrow\downarrow\uparrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} - Q^{\downarrow\uparrow\downarrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} - Q^{\downarrow\downarrow\downarrow\uparrow}_{\uparrow\downarrow\downarrow\downarrow} \\ &= -15\mathbf{z}^2w(w^2-wq^2\mathbf{c}+q) = -15\mathbf{z}^2w(w-q^2\mu^{-1})(w-q^2\mu) \\ Q_{3,4} &= Q^{\uparrow\downarrow\downarrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} - Q^{\downarrow\downarrow\uparrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} \pm \mathrm{i}(Q^{\downarrow\uparrow\downarrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} - Q^{\downarrow\downarrow\downarrow\uparrow}_{\uparrow\downarrow\downarrow\downarrow}) \\ &= -3\mathbf{z}^2w\{w^2(2\mp\sqrt{3})+wq^2\mathbf{c}+q(2\pm\sqrt{3})\}. \end{aligned} \tag{161}$$

In order to verify the functional equation (24), we also need to compute the corresponding eigenvalues of the transfer matrix (3). The relevant matrix elements are

$$T^{\uparrow\downarrow\downarrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} = a^3b+ab^3 \qquad T^{\downarrow\uparrow\downarrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} = b^2cc' \qquad T^{\downarrow\downarrow\uparrow\downarrow}_{\uparrow\downarrow\downarrow\downarrow} = abcc' \qquad T^{\downarrow\downarrow\downarrow\uparrow}_{\uparrow\downarrow\downarrow\downarrow} = a^2cc' \tag{162}$$

from which one calculates the eigenvalues

$$\begin{aligned} T_1 &= a^3b+ab^3+(a^2+ab+b^2)cc' = b(b^2+1)+(b^2+b+1)cc' \\ T_2 &= a^3b+ab^3-(a^2-ab+b^2)cc' = b(b^2+1)-(b^2-b+1)cc' \\ T_{3,4} &= a^3b+ab^3\mp(\mathrm{i}a^2\pm ab-\mathrm{i}b^2)cc' = b(b^2+1)\pm\mathrm{i}(b^2\pm\mathrm{i}b-1)cc'. \end{aligned} \tag{163}$$

One now verifies for this example that the functional equation

$$Q(z)T(z) = \phi_1(z)^4Q'(zq^2) + \phi_2(z)^4Q''(zq^{-2})$$

with $\phi_1 = bq$, $\phi_2 = aq^{-1}$, $\mu' = q\mu$, $\mu'' = \mu q^{-1}$ is valid. Let us do this explicitly for the first eigenvalues Q_1, T_1. Using (161), we write down the functional relation

$$\begin{aligned} z(z+q^2)(z+q^2\mu^2)T_1(z) &= \phi_1(z)^4zq^2(zq^2+q^2)(zq^2+q^4\mu^2) \\ &\quad + \phi_2(z)^4zq^{-2}(zq^{-2}+q^2)(zq^{-2}+\mu^2) \end{aligned} \tag{164}$$

which yields in accordance with (163) the eigenvalue

$$T_1(z) = b(z)^4q\frac{z+1}{z+q^2} + a(z)^4q\frac{z+q}{z+q^2}.$$

Note that the factors in (164) which contain zeros depending on μ cancel on both sides of the equation. We are left with one Bethe root $z^B = -q^2$ which upon the identification $z = \mathrm{e}^uq^{-1}$, $q = \mathrm{e}^{\mathrm{i}\gamma}$ is seen to trivially fulfil the Bethe ansatz equations (2),

$$\left(\frac{a(z^B)}{b(z^B)}\right)^4 = \left(\frac{1-z^Bq^2}{q-z^Bq}\right)^4 = 1.$$

The remaining eigenvalues work out in a similar manner. We now turn to the spin-sector $S^z = 0$ where we can compare our expressions for the auxiliary matrices with the one found by Baxter (see (18)).

6.2.2. $S^z = 0$. The spin-sector is six-dimensional and the non-vanishing matrix elements of (106) are computed to

$$
\begin{aligned}
m_1 &= Q^{\downarrow\downarrow\uparrow\uparrow}_{\downarrow\downarrow\uparrow\uparrow} = Q^{\downarrow\uparrow\downarrow\uparrow}_{\downarrow\uparrow\downarrow\uparrow} = Q^{\downarrow\uparrow\uparrow\downarrow}_{\downarrow\uparrow\uparrow\downarrow} = Q^{\uparrow\downarrow\downarrow\uparrow}_{\uparrow\downarrow\downarrow\uparrow} = Q^{\uparrow\downarrow\uparrow\downarrow}_{\uparrow\downarrow\uparrow\downarrow} = Q^{\uparrow\uparrow\downarrow\downarrow}_{\uparrow\uparrow\downarrow\downarrow} = \operatorname{Tr} A^2 D^2 \\
&= 3\mathbf{z}^2(qw^4 + 4q^2w^2 + 1) \\
m_2 &= Q^{\downarrow\uparrow\downarrow\uparrow}_{\downarrow\downarrow\uparrow\uparrow} = Q^{\downarrow\uparrow\uparrow\downarrow}_{\downarrow\uparrow\downarrow\uparrow} = Q^{\uparrow\downarrow\downarrow\uparrow}_{\downarrow\uparrow\downarrow\uparrow} = Q^{\uparrow\downarrow\uparrow\downarrow}_{\downarrow\uparrow\uparrow\downarrow} = Q^{\downarrow\uparrow\downarrow\uparrow}_{\uparrow\uparrow\downarrow\downarrow} = Q^{\uparrow\downarrow\uparrow\downarrow}_{\uparrow\downarrow\downarrow\uparrow} = Q^{\uparrow\uparrow\downarrow\downarrow}_{\uparrow\downarrow\uparrow\downarrow} = Q^{\downarrow\downarrow\uparrow\uparrow}_{\uparrow\downarrow\uparrow\downarrow} \\
&= \operatorname{Tr} ADCB = 3w\mathbf{z}^2q^2((q + w^2)q\mathbf{c} - w) = 3\mathbf{z}^2(\mathbf{c}w^3 - q^2w^2 + \mathbf{c}qw) \\
m_3 &= Q^{\downarrow\uparrow\uparrow\downarrow}_{\downarrow\downarrow\uparrow\uparrow} = Q^{\uparrow\uparrow\downarrow\downarrow}_{\downarrow\uparrow\uparrow\downarrow} = Q^{\uparrow\downarrow\downarrow\uparrow}_{\uparrow\uparrow\downarrow\downarrow} = Q^{\downarrow\downarrow\uparrow\uparrow}_{\uparrow\downarrow\downarrow\uparrow} = \operatorname{Tr} ABDC \\
&= 3\mathbf{z}^2q^2w((1 + w^2q)q\mathbf{c} + 2w) = 3\mathbf{z}^2(\mathbf{c}qw^3 + 2q^2w^2 + \mathbf{c}w) \\
m_4 &= Q^{\uparrow\downarrow\downarrow\uparrow}_{\downarrow\downarrow\uparrow\uparrow} = Q^{\downarrow\downarrow\uparrow\uparrow}_{\downarrow\uparrow\uparrow\downarrow} = Q^{\downarrow\uparrow\uparrow\downarrow}_{\uparrow\uparrow\downarrow\downarrow} = Q^{\uparrow\uparrow\downarrow\downarrow}_{\uparrow\downarrow\downarrow\uparrow} = \operatorname{Tr} ACDB \\
&= 3\mathbf{z}^2q^2w((1 + w^2)\mathbf{c} + 2w) = 3\mathbf{z}^2(\mathbf{c}q^2w^3 + 2q^2w^2 + \mathbf{c}q^2w) \\
m_5 &= Q^{\uparrow\downarrow\uparrow\downarrow}_{\downarrow\downarrow\uparrow\uparrow} = Q^{\downarrow\downarrow\uparrow\uparrow}_{\downarrow\uparrow\downarrow\uparrow} = Q^{\uparrow\uparrow\downarrow\downarrow}_{\downarrow\uparrow\downarrow\uparrow} = Q^{\downarrow\uparrow\downarrow\uparrow}_{\downarrow\uparrow\uparrow\downarrow} = Q^{\uparrow\downarrow\uparrow\downarrow}_{\uparrow\uparrow\downarrow\downarrow} = Q^{\uparrow\downarrow\downarrow\uparrow}_{\uparrow\downarrow\uparrow\downarrow} = Q^{\downarrow\uparrow\uparrow\downarrow}_{\uparrow\downarrow\uparrow\downarrow} = Q^{\downarrow\uparrow\downarrow\uparrow}_{\uparrow\downarrow\downarrow\uparrow} \\
&= \operatorname{Tr} ADBC = m_2 \\
m_6 &= Q^{\uparrow\uparrow\downarrow\downarrow}_{\downarrow\downarrow\uparrow\uparrow} = Q^{\uparrow\downarrow\downarrow\uparrow}_{\downarrow\uparrow\uparrow\downarrow} = Q^{\downarrow\downarrow\uparrow\uparrow}_{\uparrow\uparrow\downarrow\downarrow} = Q^{\downarrow\uparrow\uparrow\downarrow}_{\uparrow\downarrow\downarrow\uparrow} = \operatorname{Tr} B^2C^2 \\
&= 3\mathbf{z}^2q^2w^2(\mathbf{c}^2 - 2 - q - q^{-1}) \\
m_7 &= Q^{\uparrow\downarrow\uparrow\downarrow}_{\downarrow\uparrow\downarrow\uparrow} = Q^{\downarrow\uparrow\downarrow\uparrow}_{\uparrow\downarrow\uparrow\downarrow} = \operatorname{Tr} BCBC \\
&= 3\mathbf{z}^2q^2w^2(\mathbf{c}^2 + 2).
\end{aligned}
\tag{165}
$$

Under the appropriate choice of basis the auxiliary matrix is

$$
Q|_{S^z=0} = \begin{pmatrix}
m_1 & m_2 & m_3 & m_4 & m_5 & m_6 \\
m_5 & m_1 & m_2 & m_2 & m_7 & m_5 \\
m_4 & m_5 & m_1 & m_6 & m_2 & m_3 \\
m_3 & m_5 & m_6 & m_1 & m_2 & m_4 \\
m_2 & m_7 & m_5 & m_5 & m_1 & m_2 \\
m_6 & m_2 & m_4 & m_3 & m_5 & m_1
\end{pmatrix}
$$

and has the six eigenvalues

$$
\begin{aligned}
Q_1 &= m_1 - m_7 = 3\mathbf{z}^2\{qw^4 - (\mathbf{c}^2 - 2)q^2w^2 + 1\} = 3\mathbf{z}^2q(w^2 - q\mu^2)(w^2 - q\mu^{-2}) \\
Q_2 &= m_1 + m_6 - m_3 - m_4 \\
&= 3\mathbf{z}^2\{qw^4 - q\mathbf{c}(1 + q)w^3 + (\mathbf{c}^2 - 2 - q - q^2)q^2w^2 - \mathbf{c}(1 + q^2)w + 1\} \\
Q_{3,4} &= m_1 - m_6 \pm \mathrm{i}(m_3 - m_4) \\
&= 3\mathbf{z}^2\{qw^4 \pm \mathrm{i}q\mathbf{c}(1 - q)w^3 + (6 - \mathbf{c}^2 + q + q^2)q^2w^2 \pm \mathrm{i}\mathbf{c}(1 - q^2)w + 1\} \\
Q_{5,6} &= \tfrac{1}{2}\left(2m_1 + m_3 + m_4 + m_6 + m_7 \pm \sqrt{32m_2^2 + (m_3 + m_4 + m_6 - m_7)^2}\right).
\end{aligned}
\tag{166}
$$

Again we see that the matrix elements as well as the eigenvalues of the auxiliary matrix only depend on the central elements of the quantum group.

In comparison, we obtain from Baxter's formula (18) the matrix elements

$$
m_1^{\mathrm{Bax}} = 1\big/m_7^{\mathrm{Bax}} = 1\big/m_6^{\mathrm{Bax}} = zq \qquad m_2^{\mathrm{Bax}} = m_5^{\mathrm{Bax}} = 1 \qquad m_3^{\mathrm{Bax}} = 1\big/m_4^{\mathrm{Bax}} = q^{-1}.
$$

In order to match the different conventions in the choice of Boltzmann weights, one has to multiply by the additional normalization factor $\tilde{\rho} = (zq)^{\frac{M}{4}}$ (see [27]). One then finds the eigenvalues

$$
\begin{aligned}
Q_1^{\mathrm{Bax}} &= \tilde{\rho}(zq - 1/(zq)) = z^2q^2 - 1 \\
Q_2^{\mathrm{Bax}} &= \tilde{\rho}(zq + z^{-1}q^2 - q^2 - q) = q^2(z^2 - (1+q)z + q) \\
Q_{3,4}^{\mathrm{Bax}} &= \tilde{\rho}(zq - z^{-1}q^2 \pm i(q^2 - q)) = q^2(z^2 \mp i(1-q)z - q) \\
Q_{5,6}^{\mathrm{Bax}} &= \tfrac{1}{2}\big(2z^2q^2 + (1+q^2)z + 2 \pm z\sqrt{32q^2 + (1+q^2)^2}\big).
\end{aligned}
\tag{167}
$$

In section 5, we derived from (106) solutions to Baxter's functional equation (9) by summing over all the points in one fibre of the hypersurface (cf (119)). Let us check for $s = 0$ whether those solutions match (18) up to a possible normalization factor. As all the auxiliary matrices in the same fibre commute with each other, we can simply sum up the eigenvalues. Let us do this for the first four eigenvalues,

$$
\begin{aligned}
Q_1^{(s)} &= \sum_{\ell=0}^{2} Q_{1,\ell} = 3\mathbf{z}^2 q(z^2 - q)\sum_{\ell=0}^{2} q^{-\ell s}\{q^{-\ell}\mu^{-4}z^2 - q\} \underset{s=0}{\equiv} -9\mathbf{z}^2q^2(z^2 - q) \\
Q_2^{(s=0)} &= \sum_{\ell=0}^{2} Q_{2,\ell} = 9\mathbf{z}^2q^2\{z^2 - (1+q)z + q\} \\
Q_{3,4}^{(s=0)} &= \sum_{\ell=0}^{2} Q_{3,4,\ell} = -9\mathbf{z}^2q^2\{z^2 \pm \mathrm{i}(1-q)z - q\}.
\end{aligned}
\tag{168}
$$

For the fifth and sixth eigenvalue, it has been checked numerically for several values that the eigenvalues $Q_{5,6}^{(s=0)}$ are non-zero as well. Note that the polynomials for the second as well as the third and fourth eigenvalue are the same as in (167), whence we obtain the same Bethe roots and eigenvalues for the transfer matrix.

As before we can now check that the eigenvalues of the transfer matrix are correctly obtained form the functional relation. Let us perform this consistency check for the first eigenvalue in (166). The non-vanishing matrix elements of the transfer matrix are

$$
\begin{aligned}
&T^{\downarrow\downarrow\uparrow\uparrow}_{\downarrow\downarrow\uparrow\uparrow} = T^{\downarrow\uparrow\downarrow\uparrow}_{\downarrow\uparrow\downarrow\uparrow} = T^{\downarrow\uparrow\uparrow\downarrow}_{\downarrow\uparrow\uparrow\downarrow} = T^{\uparrow\downarrow\downarrow\uparrow}_{\uparrow\downarrow\downarrow\uparrow} = T^{\uparrow\downarrow\uparrow\downarrow}_{\uparrow\downarrow\uparrow\downarrow} = T^{\uparrow\uparrow\downarrow\downarrow}_{\uparrow\uparrow\downarrow\downarrow} = 2a^2b^2 \\
&T^{\downarrow\uparrow\downarrow\uparrow}_{\downarrow\downarrow\uparrow\uparrow} = T^{\downarrow\uparrow\uparrow\downarrow}_{\downarrow\uparrow\downarrow\uparrow} = T^{\uparrow\downarrow\downarrow\uparrow}_{\downarrow\uparrow\downarrow\uparrow} = T^{\uparrow\downarrow\uparrow\downarrow}_{\downarrow\uparrow\uparrow\downarrow} = T^{\downarrow\uparrow\downarrow\uparrow}_{\uparrow\uparrow\downarrow\downarrow} = T^{\uparrow\downarrow\uparrow\downarrow}_{\uparrow\downarrow\downarrow\uparrow} = T^{\uparrow\uparrow\downarrow\downarrow}_{\uparrow\downarrow\uparrow\downarrow} = T^{\downarrow\downarrow\uparrow\uparrow}_{\uparrow\downarrow\uparrow\downarrow} = abcc' \\
&T^{\downarrow\uparrow\uparrow\downarrow}_{\downarrow\downarrow\uparrow\uparrow} = T^{\uparrow\uparrow\downarrow\downarrow}_{\downarrow\uparrow\uparrow\downarrow} = T^{\uparrow\downarrow\downarrow\uparrow}_{\uparrow\uparrow\downarrow\downarrow} = T^{\downarrow\downarrow\uparrow\uparrow}_{\uparrow\downarrow\downarrow\uparrow} = a^2cc' \\
&T^{\uparrow\downarrow\downarrow\uparrow}_{\downarrow\downarrow\uparrow\uparrow} = T^{\downarrow\downarrow\uparrow\uparrow}_{\downarrow\uparrow\uparrow\downarrow} = T^{\downarrow\uparrow\uparrow\downarrow}_{\uparrow\uparrow\downarrow\downarrow} = T^{\uparrow\uparrow\downarrow\downarrow}_{\uparrow\downarrow\downarrow\uparrow} = b^2cc' \\
&T^{\uparrow\downarrow\uparrow\downarrow}_{\downarrow\downarrow\uparrow\uparrow} = T^{\downarrow\downarrow\uparrow\uparrow}_{\downarrow\uparrow\downarrow\uparrow} = T^{\uparrow\uparrow\downarrow\downarrow}_{\downarrow\uparrow\downarrow\uparrow} = T^{\downarrow\uparrow\downarrow\uparrow}_{\downarrow\uparrow\uparrow\downarrow} = T^{\uparrow\downarrow\uparrow\downarrow}_{\uparrow\uparrow\downarrow\downarrow} = T^{\uparrow\downarrow\downarrow\uparrow}_{\uparrow\downarrow\uparrow\downarrow} = T^{\downarrow\uparrow\uparrow\downarrow}_{\uparrow\downarrow\uparrow\downarrow} = T^{\downarrow\uparrow\downarrow\uparrow}_{\uparrow\downarrow\downarrow\uparrow} = abcc' \\
&T^{\uparrow\downarrow\uparrow\downarrow}_{\downarrow\uparrow\downarrow\uparrow} = T^{\downarrow\uparrow\downarrow\uparrow}_{\uparrow\downarrow\uparrow\downarrow} = (cc')^2.
\end{aligned}
\tag{169}
$$

From these identities, one calculates the eigenvalues

$$
\begin{aligned}
T_1 &= 2(ab)^2 - (cc')^2 \\
T_2 &= 2(ab)^2 - (a^2 + b^2)cc' \\
T_{3,4} &= 2(ab)^2 \pm \mathrm{i}(a^2 - b^2)cc' \\
T_{5,6} &= \tfrac{1}{2}\left(4(ab)^2 + cc'(a^2 + b^2 + cc') \pm cc'\sqrt{32(ab)^2 + (a^2 + b^2 - cc')^2}\right).
\end{aligned}
\tag{170}
$$

The functional relation

$$Q(z)T(z) = \phi_1(z)^4 Q'(zq^2) + \phi_2(z)^4 Q''(zq^{-2})$$

with $\phi_1 = bq, \phi_2 = aq^{-1}, \mu' = q\mu, \mu'' = \mu q^{-1}$ implies for the eigenvalues T_1, Q_1 the identity (setting $\rho = 1$ in (5)),

$$T_1(z) = 2b(z)^2 - c(z)^2c'(z)^2 = \phi_1(z)^4\frac{z^2q - q}{z^2 - q} + \phi_2(z)^4\frac{z^2q^2 - q}{z^2 - q}.$$

After a short calculation, the above equation is shown to be true. The corresponding Bethe roots of the eigenvalue are easily deduced from (166) to be $z^B_\pm = \pm q^2$ which are easily seen to solve the Bethe ansatz equations (2) when setting $z = \mathrm{e}^u q^{-1}$, $q = \mathrm{e}^{\mathrm{i}\gamma}$,

$$\left(\frac{a(z_\pm)}{b(z_\pm)}\right)^4 = \left(\frac{1 - z_\pm q^2}{q - z_\pm q}\right)^4 = \frac{z_\mp/z_\pm - q^2}{z_\mp/z_\pm q^2 - 1} = 1.$$

For the remaining eigenvalues the functional relation has been checked numerically.

7. Conclusions

Starting from evaluation, representations of the quantum loop algebra $U_q(\widetilde{sl}_2)$ families of auxiliary matrices for the six-vertex model at roots of unity have been explicitly constructed. See the definitions (68), (71), (106) and apply the representation (43). For odd roots of unity, the auxiliary matrices depend on three, for even roots of unity on one additional parameter besides the spectral variable z and the deformation parameter q. In comparison to earlier results in the literature, the auxiliary matrices (106) have several advantages. They extend to all spin-sectors and do not contain formal power series since the auxiliary space can be kept finite-dimensional at roots of unity. They have been demonstrated to be of the simple form (16) and to satisfy the functional equation (24) which can be interpreted in terms of representation theory (cf (25)). All operators in this functional equation have been shown to commute with each other (cf (144)), whence the eigenvalues of the transfer matrix (3) and the Bethe ansatz equations (2) can be derived. In the present paper, this has been done for the two simple examples $N = 3$, $M = 3, 4$. In a forthcoming paper [23], the eigenvalues of the constructed auxiliary matrices will be investigated for general N, M.

7.1. The geometric interpretation

Applying the concept of evaluation representations of $U_q(\widetilde{sl}_2)$ allowed for a simple classification of the auxiliary matrices and a geometric interpretation of their parameters. Regardless which root-of-unity representation is used to write down solutions for the L-operator (71) the final auxiliary matrix (106) only depends on the values of the central elements of the quantum algebra. Employing the results of [20, 21], these values were shown to specify points on a three-dimensional complex hypersurface Spec Z defined in (36).

In fact, the presented construction of auxiliary matrices can be interpreted as the definition of the following map from the direct product of the complex numbers with Spec Z into the operator space over the spin-chain:

$$Q : \mathbb{C} \times (\text{Spec}\, Z \backslash D) \to \text{End}\big(\pi_1^{0\otimes M}\big) \qquad p = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{c} = \mu + \mu^{-1}) \to Q_p(z) \tag{171}$$

with $Q_p(z)$ given by (106). The hypersurface Spec $Z \backslash D$ is an N-fold fibration and under multiplication with the transfer matrix the auxiliary matrix is shifted to the neighbouring points

$$p' = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{c}' = \mu q + q^{-1}\mu^{-1}) \qquad \text{and} \qquad p'' = (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{c}'' = \mu q^{-1} + q\mu^{-1})$$

in the same fibre. If we would have allowed for points p in the discriminant set D (see (39)), the result would have been the known functional relations between transfer matrices of higher spin $1 \leqslant 2s \leqslant N - 1$ (see, e.g., [43]). The constructed map (171) allows us to carry important mathematical structures of the hypersurface Spec Z over to the family of auxiliary matrices $\{Q_p(z)\}_{p \in \text{Spec}\, Z}$.

7.1.1. The quantum coadjoint action on auxiliary matrices. The quantum coadjoint action given by the automorphisms (50) on the hypersurface (36) can be extended in a natural manner to the family of auxiliary matrices (106) setting (for N odd),

$$G \times \operatorname{Spec} Z \ni (g, p) \to g \cdot Q_p(z) := Q_{gp}(z). \tag{172}$$

As the points p, p', p'' in (24) belong to the same fibre over Spec Z_0 and the Casimir element remains invariant under the group action, the functional equation is preserved. The definition (172) is a manifestation of the infinite-dimensional symmetry of the six-vertex model at roots of unity. Since the six-vertex transfer matrix does not depend on the point p in the hypersurface, one might choose any element in the family of auxiliary matrices to solve the eigenvalue problem of (3) respectively (1). Therefore, one leads to look for transformations in the parameters p which leave the set of auxiliary matrices invariant. These transformations are precisely given by the infinite-dimensional group G. Another way of expressing the symmetry is by observing that for any group element $g \in G$, one has

$$[T(z), Q_{gp}(w)] = 0 \qquad \forall g \in G \qquad z, w \in \mathbb{C}. \tag{173}$$

Since the auxiliary matrices belonging to different fibres in the hypersurface do not commute in general, the vanishing of the above commutator exhibits infinitely many non-Abelian conserved quantities defined for all spin-sectors.

Mathematically, the quantum coadjoint action is interesting since it allows to explore the structure of the base space Spec Z_0 which can be given the structure of a Poisson–Lie group [20, 21]. In order to connect these mathematical structures with the eigenvalue problem of the transfer matrix further investigations are needed. In particular, it would be helpful to have a direct implementation of the group action (172) on the spin-chain. This would allow one to obtain further insight how the earlier observed loop symmetry in the commensurate sectors $S^z = 0 \bmod N$ [6, 12, 13] extends to the non-commensurate ones. It is an intriguing observation that the generators of the quantum coadjoint action are closely related to the restricted quantum group expressing the loop algebra symmetry. This aspect will be subject to future investigations.

7.2. Broken symmetries

Besides the infinite-dimensional symmetry, the auxiliary matrices also break the finite symmetries (7) and (6) of the six-vertex model. (Note that all three of these symmetries have also been implemented as mappings on the hypersurface Spec Z.) This shows that the present setting is more general than the coordinate space Bethe ansatz. Spin-conservation, which is essential for the application of the coordinate space Bethe ansatz, can be restored by taking the nilpotent limit. That is, for the following subvariety of auxiliary matrices, one has

$$\left[Q_{p_\mu}(z), S^z\right] = 0 \qquad p_\mu = (0, 0, \mu^{N'}, \mu + \mu^{-1}) \qquad \mu \in \mathbb{C}^\times. \tag{174}$$

However, the action (172) of the infinite-dimensional automorphism group G on this one-parameter family forces one to consider also cyclic representations which do not preserve the total spin and violate the conditions (11) and (12). This explicitly shows the statement made in the introduction that the full symmetry present at odd roots of unity becomes manifest when the coordinate space Bethe ansatz ceases to be applicable.

For even roots of unity cyclic representations had to be excluded since an intertwiner does not exist. Nevertheless, the hypersurface, the decomposition of the tensor product via the exact sequence (25) and the quantum coadjoint action are equally well defined. It would be interesting to find the construction for cyclic representations also in this case. The functional relation (128) derived for even roots of unity with N' odd might serve as a starting point.

Another way to proceed is to exploit the quantum coadjoint action (172) for even roots of unity. Again this subject is left to future investigations.

Within the framework of the coordinate space Bethe ansatz spin-reversal symmetry is broken in the sectors $S^z \neq 0$. The auxiliary matrices constructed in this work also break spin-reversal symmetry (cf (130)). For example, one has for the above one-parameter family $Q_\mu \equiv Q_{p_\mu}$ the transformation

$$\mathfrak{R} Q_\mu(z) \mathfrak{R} = Q_{\mu^{-1}}(z\mu^{-2}). \tag{175}$$

From this transformation law, one infers that spin-reversal symmetry is restored when $\mu \to 1$. In the construction presented here this amounts to choosing a reducible representation in the definition of (22). This transformation behaviour which has been derived from the algebra automorphism (73) is different from previous constructed auxiliary matrices for the six-vertex model. Baxter's expression (18) only applies to the spin-zero sectors where it does not break spin-reversal invariance in accordance with the Bethe ansatz. The auxiliary matrix considered in [27] breaks spin-reversal symmetry outside the sectors $S^z = 0$ due to an additional factor $s_0^{S^z}$, $s_0 \in \mathbb{C}$ in the 'constant' (26).

7.3. Outlook

The difference between the two construction procedures for auxiliary matrices in the literature needs to be further clarified. We explicitly verified for the $S^z = 0$ sector of the four-chain that the auxiliary matrices constructed here are different form Baxter's expression (18).

For the auxiliary matrices (106), this is to be expected as Baxter's functional equation (9) is formulated in terms of a single auxiliary matrix, while (24) obtained from representation theory involves three different ones.

This discrepancy between the functional equations is removed when summing the auxiliary matrices (106) over the points in a single fibre. (In the case of even roots of unity, one has to sum over two fibres.) One then obtains the solutions (119) to Baxter's functional equation (9) which are defined on the base manifold Spec Z_0 (cf (37)).

The solutions (119) might be singular matrices in general, although for the spin-zero sector of the four-chain this was not the case. Singular matrices were also encountered for the auxiliary matrix (106) in the case of the three-chain. Whether these singularities persist for longer spin-chains needs to be numerically investigated. This is of particular importance in order to clarify whether all eigenvalues of the transfer matrix can be obtained using the method of auxiliary matrices. It would also shed further light on the differences between the two different construction methods.

In this context, it would also be of particular interest to make contact with the results in [27] for $q^N \neq 1$ and investigate further the implications of the formal power series (26). This would be another step forward to understand the representation theoretic meaning of the Bethe ansatz solutions for all values of the deformation parameter q.

The construction procedure for auxiliary matrices at roots of unity presented here can be generalized to higher spin and higher rank. While the analysis of the irreducible representations at roots of unity has been carried out for all simple quantum Lie algebras [20, 21], the crucial input needed is the existence of an evaluation homomorphism (53). The latter allowed us to make contact with the corresponding quantum loop algebra underlying the respective trigonometric vertex model. Such evaluation homomorphisms only exist for sl_n [42]. In the case of the other algebras, it might depend on the specific nature of the evaluation representation whether one finds analogous results.

The other generalization which comes to mind is the connection with elliptic models, most of all the eight-vertex model which historically has been the starting point for the observation of extra symmetries at roots of unity. Fabricius and McCoy observed [44] that the eigenvalues of the eight-vertex auxiliary matrix constructed by Baxter [15] satisfy a functional equation at roots of unity which does not involve the transfer matrix. This functional equation in conjunction with sum rules for the Bethe roots allowed them to calculate the dimension of the degenerate eigenspaces of the transfer matrix. The formulation of an analogous functional equation for the six-vertex model is an open problem.

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